RELAXATION PROCESSES IN COMPLETELY IONIZED PLASMA IN GENERALIZED LORENTZ MODEL

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On the basis of the Landau kinetic equation a generalized Lorentz model is proposed, which contrary to the standard model, considers ion system as an equilibrium one. For electron system kinetic equation of the Fokker-Planck type is obtained. In the Bogolyubov method of the reduced description, which is based on his idea of the functional hypothesis, basic equations for electron hydrodynamics construction with account for temperature and macroscopic velocity relaxation processes (kinetic modes of the system) is elaborated. The obtained equations are analyzed near the end of the relaxation processes when the theory has an additional small parameter. The main in small gradients approximation is studied in details, it corresponds to the description of relaxation processes in a spatially uniform case. The obtained equations are approximately solved by the method of truncated expansion in the Sonine polynomials. The velocity and temperature relaxation coefficients are discussed in one- and two-polynomial approximation. As a result the relaxation coefficients are calculated in one-polynomial approximation.

Keywords: method of the reduced description, generalized Lorentz model, kinetic modes, plasma hydrodynamics, relaxation coefficients, one-polynomial approximation.

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1. Introduction

Investigation of nonequilibrium processes in a system with account of relaxation processes is an actual modern problem of kinetic theory (see [1, 2] and also a review [3]). Relaxation processes in the narrow sense of the word mean processes that can be observed in spatially uniform states of a system too. In the vicinity of the equilibrium these processes are described by kinetic modes of the system. The problem of their investigation is that the theory has not a small parameter, which could help to investigate these processes. The most important problem of such type is the description of states that precede in time to states in which relaxation processes already have finished. In fact, we are talking about the formation of well-studied nonequilibrium states in the process of evolution. In particular, in a many-component system, for example in plasma, we speak about the processes of alignment of macroscopic velocities and temperature of the components. The pioneering study of such processes was carried out by Landau [4] on the basis of the idea that the local equilibrium state is quickly established in the components of the system (see also the known paper [5]). Another important class of tasks is the problem of dissipative flow formation in hydrodynamics. The pioneering study of such processes was performed by Grad [6] on the basis of using nonequilibrium distribution functions which are taken in the form of truncated expansion in the Hermite orthogonal polynomials. A certain step forward in the study of relaxation processes was our idea of investigating them in the vicinity of the completion of these processes with introducing a new small parameter in the theory (see review [3]). This method has been used in a number of papers for the study of completely ionized plasma [7], subsystems of polarons and phonons of solids [8].

The complexity of such systems is a certain obstacle in this research. That is why in this paper we deal with introducing a generalization of the Lorentz model (see, for example, [9]) and advancing the research into the role of relaxation processes in spatially non-uniform plasma in the framework of this model.

The work is structured as follows. Section 2 formulates a generalized Lorentz model. In Section 3 based on the Bogolyubov reduced description method (see review [5]) the basic equations of hydrodynamics of the electron subsystem of plasma are formulated with taking
into account relaxation processes. In Section 4 a perturbation theory is developed to study the spatially uniform states of the system at the end of the relaxation processes. In Section 5 we are talking about an approximate solution of the integral equations of the theory by the method of the truncated expansion in Sonine polynomials. In Section 6 the coefficients of electron temperature and velocity relaxation are calculated.

2. Generalized Lorentz model

Completely ionized electron-ion plasma is investigated. The Landau kinetic equation in the presence of a constant homogeneous external electric field \( E \) lies in the basis of our consideration

\[
\frac{\partial f_{ap}(x,t)}{\partial t} = -\frac{p_n}{m_n} \frac{\partial f_{ap}(x,t)}{\partial x_n} - e E_n \frac{\partial f_{ap}(x,t)}{\partial p_n} = I_{ap}(f(x,t)).
\]

(1)

Its collision integral is given by the formula (see, for example, \([4, 10]\))

\[
I_{ap}(f) = 2\pi e^2 L \sum_b \int d^3 p \left\{ \frac{\partial f_{ap}}{\partial p_{al}} - \frac{\partial f_{ap'}}{\partial p_l'} \right\} S_{al} \left( \frac{p}{m_n} - \frac{p'}{m_b} \right)
\]

(2)

where

\[
S_{al}(u) \equiv (u^2 \delta_{al} - u_a u_l) / u^3
\]

(3)

and \( L \) is a constant which is called the Coulomb logarithm. Indices \( a, b \) run meanings \( e, i \) indicating electron and ion components; \( e_e = -e \), \( e_i = e z \) ( \( e > 0 \), \( z \) is the charge number of the ion). The component distribution function \( f_{ap}(x,t) \) is normalized by the condition

\[
\int d^3 p f_{ap}(x,t) = n_a(x,t)
\]

(4)

where \( n_a(x,t) \) are densities of number of particles of components.

Further we will assume that ions form an equilibrium system in a state of rest with temperature \( T_0 \)

\[
f_{ip}(x,t) = f_{ip}^{eq}, \quad f_{ip}^{eq} = w_{ip}, \quad w_{ip} \equiv \frac{n_0}{(2\pi M T_0)^{3/2}} e^{-\frac{p^2}{2 M T_0}}.
\]

(5)

(\( n_0 \) is ion density, \( M \) is an ion mass). The ion-ion and electron-electron interaction are neglected, that is, we introduce a generalized Lorentz model. Electron distribution function is denoted here by \( f_p(x,t) \). The kinetic equation for this function takes the form

\[
\frac{\partial f_p(x,t)}{\partial t} = -\frac{p_n}{m} \frac{\partial f_p(x,t)}{\partial x_n} + e E_n \frac{\partial f_p(x,t)}{\partial p_n} + I_p(f(x,t))
\]

where the ion-electron collision integral is given by the formula

\[
I_p(f) = 2\pi e^2 z^2 L \int d^3 p \left\{ \frac{\partial f_p}{\partial p_{al}} w_{ip} - \frac{\partial w_{ip}}{\partial p_l'} \right\} S_{al} \left( \frac{p}{m} - \frac{p'}{M} \right).
\]

(7)

(\( m \) is electron mass). Let us introduce the notation

\[
D_{al}(p) = 2\pi \alpha \int d^3 p w_{ip} S_{al} \left( \frac{p}{m} - \frac{p'}{M} \right), \quad \alpha = e^4 z^2 L
\]

(8)

and take into account that
Relaxation processes in completely ionized plasma in generalized Lorentz model

\[ \int d^3 p' \frac{\partial w_{ip'}}{\partial p_i'} S_{nl} \left( \frac{p}{m} - \frac{p'}{M} \right) = - \int d^3 p' w_{ip'} \frac{\partial}{\partial p_i'} S_{nl} \left( \frac{p}{m} - \frac{p'}{M} \right) = \]

\[ = \frac{m}{M} \frac{\partial}{\partial p_i} \int d^3 p' w_{ip'} S_{nl} \left( \frac{p}{m} - \frac{p'}{M} \right). \]

Then collision integral (7) can be written in the form

\[ I_p(f) = - \frac{\partial}{\partial p_i} \left( A_n(p)f_p - D_{nl}(p) \frac{\partial f_p}{\partial p_i} \right) \]  

(9)

where we denote

\[ A_n(p) = \frac{m}{M} \frac{\partial D_{nl}(p)}{\partial p_i}. \]

(10)

We obtain a kinetic equation of the Fokker–Planck type. According to standard terms, quantities \( D_{nl}(p), A_n(p) \) are called the diffusion coefficient in the momentum space and the friction force (see about this class of equations, for example, in [9, 10]).

Note that in equilibrium the distribution function of electrons \( f_p \) is given by the Maxwell distribution

\[ f_p^{eq} = w_p, \quad w_p = \frac{n}{(2\pi m T_0)^{3/2}} e^{-\frac{p^2}{2m T_0}} \]

(11)

because according to (7) the formula

\[ I_p(w) = 0 \]

(12)

is true as a consequence of the identity \( u_i S_{nl}(u) = 0 \). In terms of the notation (9) it means that

\[ A_n(p) = -D_{nl}(p) \frac{p_i}{m T_0} \]

(13)

is true. So, finally the collision integral (9) takes the form

\[ I_p(f) = \frac{\partial}{\partial p_i} \left( D_{nl}(p) \left( \frac{\partial f_p}{\partial p_i} + \frac{p_i}{m T_0} f_p \right) \right) \]

(14)

The function \( D_{nl}(p) \) defined in (8) has a fairly simple structure, which simplifies our further research. In particular, it has a simple dependence on dimensional quantities. This allows us to find out the structure of dependence on the dimensional values of all objects of the theory.

3. Basic equations of the reduced description method for electron hydrodynamics

Let us construct the basic equations of hydrodynamics in the presence of relaxation processes of the component temperature and velocity equalizing in the generalized Lorentz model. We proceed from the mass, energy, and momentum conservation laws, which, in accordance with the kinetic equation (6), (14), take the form

\[ \frac{\partial \sigma(x,t)}{\partial t} = - \frac{\partial \pi_n(x,t)}{\partial x_n}, \quad \frac{\partial \pi_n(x,t)}{\partial t} = \frac{\partial t_{nl}(x, \tilde{f}(t))}{\partial x_n} - \frac{e}{m} E_f \sigma(x,t) + R_f(x, \tilde{f}(t)), \]

\[ \frac{\partial \sigma(x,t)}{\partial t} = - \frac{\partial q_n(x, \tilde{f}(t))}{\partial x_n} - \frac{e}{m} E_n \pi_n(x,t) + R_0(x, \tilde{f}(t)) \]

(15)
where mass $\sigma(x,t)$, momentum $\pi_j(x,t)$ and energy $\varepsilon(x,t)$ densities according to (4) are defined by the formulas

$$
\sigma(x,t) = m \int d^3 p f_p^j(x,t), \quad \pi_j(x,t) = \int d^3 p p_j f_p^j(x,t), \\
\varepsilon(x,t) = \int d^3 p e f_p^j(x,t), \quad (e_p = p^2 / 2m) \tag{16}
$$

(along with the mass density it is convenient to use the density of the number of particles $n = \sigma / m$). Energy $q_m(x,f)$ and momentum $t_m(x,f)$ flux densities and sources $R_0(x,f)$ and $R_l(x,f)$ are given by the expressions

$$
t_m(x,f) = \frac{1}{m} \int d^3 p p_j f_p^j(x), \quad q_m(x,f) = \frac{1}{m} \int d^3 p e f_p^j(x); \\
R_0(x,f) = \int d^3 p e f_p^j(x), \quad R_l(x,f) = \int d^3 p p_j f_p^j(x,f). \tag{17}
$$

Electron component temperature $T(x,t)$ and velocity $u_n(x,t)$ are the basic hydrodynamic parameters and defined by usual formulas

$$
\varepsilon(x,t) = \frac{3}{2m} \sigma(x,t) T(x,t) + \frac{1}{2} \sigma(x,t) u(x,t)^2, \quad \pi_j(x,t) = \sigma(x,t) u_j(x,t) \tag{18}
$$

because electron-electron interaction is neglected. They are based on the local equilibrium idea according to which

$$
\varepsilon = \int d^3 p e f_p^j w_{p-mu}, \quad \pi_j = \int d^3 p p_j f_p^j \tag{19}
$$

where the Maxwell distribution $w_p$ defined by (11) is used.

Equations (15) give the equations of hydrodynamics if we express their right-hand sides via independent variables (parameters of the reduced description)

$$
\xi_0(x,t): \quad \xi_0(x,t) \equiv T(x,t), \quad \xi_1(x,t) \equiv u_1(x,t), \quad \xi_3(x,t) \equiv \sigma(x,t), \tag{20}
$$

which describe a state of the system in hydrodynamics. This can be done through placing the Bogolyubov idea of the functional hypothesis

$$
f_p^j(x,t) \rightarrow \quad \xi_0(x,t) \tag{21}
$$

in the basis of consideration. According to this idea the distribution function $f_p^j(x,t)$ becomes a functional $f_p^j(x,\xi) \equiv f_p^j(x,\xi)$ of variables $\xi_j(x)$ at the times much greater than a certain characteristic time $\tau_0$. This relation is the basis of the Bogolyubov formulation of the Chapman–Enskog method, which is a special case of his method of reduced description. In accordance with (16) and (18) the reduced description parameters are expressed in terms of some moments of the distribution function. Since the distribution function is determined by all moments, such a description is a reduced one.

In the system under consideration the electron component temperature and velocity relaxation is observed

$$
T(x,t) \rightarrow T_0, \quad u_n(x,t) \rightarrow -\kappa E_l \tag{22}
$$

where $\tau_T$ and $\tau_u$ are corresponding relaxation times, $\kappa$ is electron mobility. The time $\tau_0$ in the functional hypothesis (21) is assumed to fulfill the condition $\tau_0 \ll \tau_T, \tau_u$, hence the relaxation processes in the situation under consideration are still taking place. This does not fit into the standard Chapman–Enskog method, in which the only small parameter is
the parameter of the smallness \( g \) of the gradients of the reduced description parameters that is introduced by the estimates

\[
\partial^2 \xi_n(x) / \partial x_n \partial x_n \sim g^2, \quad g = l_0 / L.
\] (23)

Here \( l_0 \) is the mean free path, \( L \) is the characteristic size of the non-homogeneity of the electron distribution in space. Zero approximation in the gradients \( \xi_n(x) \) for the distribution function \( f_p(x, \xi) \) describes relaxation in a spatially uniform case. The distribution function \( f_p(0) \) satisfies the integral equation, whose solution cannot be found in the standard Chapman–Enskog method. In our work a new small parameter \( \varepsilon \) that allows us to calculate the distribution function \( f_p(0) \) is introduced.

As a result, the equations of hydrodynamics can be written in the form

\[
\frac{\partial \xi_n}{\partial t} = L_n(x, f(\xi(t)))
\] (24)

where \( L_n(x, f) \) is a functional of \( f_p(x) \). In accordance with (15) and (18), the exact form of these equations is given by the formulas

\[
\frac{\partial \sigma}{\partial t} = \sigma u_x, \quad \frac{\partial u_x}{\partial t} = -\frac{1}{\sigma m} \sigma u_x + \frac{1}{\sigma m} (R_l - eE),
\]

\[
\frac{\partial T}{\partial t} = \frac{3T}{\beta} - mu^2 \sigma u_x + \frac{2}{3n} \sigma x_x - \frac{2}{3n} \frac{\partial u_x}{\partial x_x} + \frac{2}{3n} (R_l - eE u_x).
\] (25)

For the distribution function at the reduced description \( f_p(x, \xi) \) the equation

\[
\sum_n \int d^3 x' \frac{\delta f_p(x, \xi)}{\delta \xi_n(x')} L_n(x', f(\xi)) = -\frac{P_n}{m} \sigma u_x + eE_a \frac{\partial f_p(x, \xi)}{\partial p_n} + I_p(x, f(\xi))
\] (26)

is obtained from the kinetic equation (6) with taking into account the functional hypothesis (22). Definitions (16) and (18) of the parameters of the reduced description \( \xi_n(x) \) give additional conditions for this equation

\[
\int d^3 p f_p(x, \xi) e_p = \frac{3}{2} n(x) T(x) + \frac{1}{2} \sigma(x) u(x)^2,
\]

\[
\int d^3 p f_p(x, \xi) p_x = \sigma(x) u_x(x), \quad \int d^3 p f_p(x, \xi) = n(x).
\] (27)

To construct the equations of hydrodynamics, it is necessary to solve equation (26) with the relations (27) in perturbation theory in gradients based on the estimations (23).

4. Construction of the perturbation theory for solving the equations of the theory

The basis of this work is the idea of studying hydrodynamics taking into account the relaxation phenomena near their completion, when the quantities \( T(x, t) - T_0 \) and \( u_n(x, t) + \mu E_n(x) \) in accordance with (23) are small

\[
\tau(x, t) \equiv T(x, t) - T_0 \sim \varepsilon, \quad u_n(x, t) + \mu E_n(x) \sim \varepsilon
\] (28)

where \( \varepsilon \) is a small parameter introduced formally. To simplify the consideration, we will assume that the electric field is weak. Therefore, according to (23) and (28) the estimates
are true. The solution of equation (26), with taking into account (27), is sought in the form of a double series in the gradients of parameters \( \xi_{\alpha \beta}(x) \) and the small parameter \( \varepsilon \)

\[
\begin{align*}
    f_p(x, \xi) &= f_p^{(0)} + \varepsilon f_p^{(1)} + O(\varepsilon^2), \\
    f_p^{(0)} &= f_p^{(0,0)} + f_p^{(0,2)} + O(\varepsilon^0), \\
    f_p^{(1)} &= f_p^{(1,0)} + f_p^{(1,1)} + O(\varepsilon^1)
\end{align*}
\]

where a quantity \( f_p^{(m,a)} \sim g^m \varepsilon^a \). Similarly the contributions of the perturbation theory to flows, sources, and collision integral \( q_i(x, f) \), \( t_{in}(x, f) \), \( R_q(x, f) \), \( R_i(x, f) \), \( I_p(x, f) \) are denoted. In this paper we restrict ourselves to considering the basic (zero) approximation in gradients, which describes the evolution of a spatially uniform system, that is, only the relaxation phenomena in the narrow sense. The results of the previous section allow us to construct a theory of relaxation processes in a spatially non-uniform case (in other words, hydrodynamics taking into account relaxation processes), it will be done in another work.

Distribution of the gradient-zero approximation \( f_p^{(0)} \), in accordance with (25) and (26), satisfies the nonlinear differential equation

\[
\begin{align*}
    \frac{\partial f_p^{(0)}}{\partial \psi} &= \frac{1}{\sigma} \left\{ R_i(f^{(0)}) - neE_i \right\} + \frac{2}{\sigma} \left\{ R_q(f^{(0)}) - R_i(f^{(0)}) u_i \right\} + eE_i \frac{\partial f_p^{(0)}}{\partial \psi} = I_p(f^{(0)}) \\
    \frac{\partial f_p^{(0)}}{\partial \psi} &= \frac{1}{\sigma} \left\{ R_i(f^{(0)}) - neE_i \right\} + \frac{2}{\sigma} \left\{ R_q(f^{(0)}) - R_i(f^{(0)}) u_i \right\} + eE_i \frac{\partial f_p^{(0)}}{\partial \psi} = I_p(f^{(0)})
\end{align*}
\]

with additional conditions

\[
\int d^3 x \rho_p(0) \psi = \frac{3}{2} nT + \frac{1}{2} \sigma u^2, \quad \int d^3 x \rho_p(0) \psi = \sigma u_i, \quad m \int d^3 x \rho_p(0) = \sigma.
\]

It was noted above that methods for solving such nonlinear equations were not developed. In this work it is proposed to investigate them near the end of the relaxation, when one can use the perturbation theory in parameter \( \varepsilon \). The contributions of the order \( \varepsilon^0 \) to (31) and (32) have the form

\[
\begin{align*}
    \frac{\partial f_p^{(0)}}{\partial \psi} &= \frac{1}{\sigma} \left\{ R_i(f^{(0)}) - neE_i \right\} + \frac{2}{\sigma} \left\{ R_q(f^{(0)}) - R_i(f^{(0)}) u_i \right\} + eE_i \frac{\partial f_p^{(0)}}{\partial \psi} = I_p(f^{(0)}) \\
    \int d^3 x \rho_p^{(0)} \psi &= \frac{3}{2} nT_0, \quad \int d^3 x \rho_p^{(0)} \psi = \sigma, \quad m \int d^3 x \rho_p^{(0)} = \sigma.
\end{align*}
\]

According to (12), the collision integral \( I_p(w) \) with the Maxwell distribution \( w_p \) defined in (11) equals to zero. The definition of sources in the conservation laws (17) shows that they with \( w_p \) also equal to zero

\[
R_q(w) = 0, \quad R_i(w) = 0.
\]
Relaxation processes in completely ionized plasma in generalized Lorentz model

\[ f^{(0,0)}_p = w_p. \]  

Simultaneously from formulas (20) we find the flows

\[ t^{(0,0)}_m = t_m(x, f^{(0,0)}) = \frac{1}{m} \int d^3 p p_l p_n f^{(0,0)}_p = n T \delta_{nl}, \]

\[ q^{(0,0)}_n = q_n(x, f^{(0,0)}) = \frac{1}{m} \int d^3 p e p_l p_n f^{(0,0)}_p = 0. \]

In the first order in the parameter \( \varepsilon \) from equations (31) and (32) taking into account (12) and (35) we obtain

\[ \frac{\partial f^{(0,1)}_p}{\partial t} - \frac{\varepsilon}{\sigma} R_{p}(f^{(0,1)}) - \frac{\partial f^{(0,1)}_p}{\partial \theta} = \frac{e E_1}{m} + \frac{2}{3n} R_0(f^{(0,1)}) + e E_1 \frac{\partial f^{(0,0)}_p}{\partial \theta} = I_p(f^{(0,1)}); \]

\[ \int d^3 p f^{(0,1)}_p v_p = \frac{2}{3} n \tau, \quad \int d^3 p f^{(0,1)}_p p_l = \sigma u_l, \quad \int d^3 p f^{(0,1)}_p = 0. \]

For reasons of rotational invariance the solution of these equations has the structure

\[ f^{(0,1)}_p = w_p \{ A_p p_l u_l + B_p \tau + C_p p_l E_1 \} \]

where \( A_p, B_p, C_p \) are some scalar functions. Substituting this function into equation (37) with account for the independence of the variables \( u_l, \tau, E_1 \) in (38) gives equations for the quantities \( A_p, B_p, C_p \) with additional conditions to them

\[ \hat{K} A_p p_l = \lambda_u A_p p_l, \quad \langle v_p A_p \rangle = \frac{3}{2} n; \quad \hat{K} B_p = \lambda_T B_p, \quad \langle v_p B_p \rangle = \frac{3}{2} n, \quad \langle B_p \rangle = 0; \]

\[ \hat{K} C_p p_l = \left\{ v A_p - \frac{\partial \delta}{\partial \theta} \right\} p_l, \quad \langle v_p C_p \rangle = 0. \]

Hereafter for an arbitrary quantity \( a_p \) we denote

\[ \langle a_p \rangle = \int d^3 p w_p a_p \]

as well as introduce coefficients \( \lambda_u, \lambda_T, \) and \( \nu \) by the formulas

\[ \lambda_u \delta_{nl} = -\frac{1}{\sigma} R_p(w A_p), \quad \lambda_T = -\frac{2}{3n} R_p(w B), \quad \nu \delta_{nl} = -\frac{1}{\sigma} R_p(w e C_p) + \frac{e}{m} \delta_{nl}. \]

For further it is convenient to use the operator \( \hat{K} \)

\[ I_p(w a) = -w_p \hat{K} a_p \]

which, according to (14), is given by the formula

\[ \hat{K} a_p = -\left( \frac{\partial}{\partial p_a} - \frac{p_a}{m T} \right) D_a(p) \frac{\partial a_p}{\partial p_l}. \]

With the help of the operator \( \hat{K} \) one can introduce the bilinear form (brackets)

\[ \langle a_p, b_p \rangle = \int d^3 p w_p a_p \hat{K} b_p = \int d^3 p w_p D_a(p) \frac{\partial a_p}{\partial p_a} \frac{\partial b_p}{\partial p_l}, \]

which has important properties

\[ \{ a_p, b_p \} = \{ b_p, a_p \}, \quad \{ a_p, a_p \} \geq 0, \quad \{ a_p, c \} = 0 \quad \text{for arbitrary functions, } \]

Sources (17) in the conservation laws (15) are expressed in terms of this form by relations

\[ \text{(45)} \]
\[ R_0(wa) = -\{v_p, a_p\}, \quad R(wa) = -\{p_t, a_p\}. \] (46)

Herewith the coefficients \( \lambda_u, \lambda_T, \) and \( \nu \) according to (42) are given by the formulas
\[
\lambda_u = \frac{1}{3\sigma} \{p_n, A_p p_n\}, \quad \lambda_T = \frac{2}{3n} \{v_p, B_p\}, \quad \nu = \frac{1}{3\sigma} \{p_t, C_p p_t\} + \frac{e}{m}.
\] (47)

The meaning of these coefficients is understood from the time equations for the reduced description parameters (28), which in the order \( g^0 e^1 \) of the perturbation theory, with taking into account (38) and (46), give
\[
\left( \frac{\partial \sigma}{\partial t} \right)^{(0,1)} = 0, \quad \left( \frac{\partial u_t}{\partial t} \right)^{(0,1)} = -\lambda_u u_l + \nu E_l, \quad \left( \frac{\partial T}{\partial t} \right)^{(0,1)} = -\lambda_T \tau.
\] (48)

Hereafter we call \( \lambda_u \) and \( \lambda_T \) the relaxation coefficients. From formulas (45) and identities
\[
\lambda_u \langle A_p^2 \rangle = \{A_p, p_t, A_p p_t\}, \quad \lambda_T \langle B_p^2 \rangle = \{B_p, B_p\},
\] (49)

which follow from equations (38) and definition (44), it is clear that these coefficients are positive. They describe the relaxation (22) with characteristic times
\[
\tau_u = \lambda_u^{-1}, \quad \tau_T = \lambda_T^{-1}.
\] (50)

Note that the relations (47) do not make the equations (39) nonlinear. Formulas (47) are the consequences of equations (39). Functions \( A_p p_t \) and \( B_p \) are the operator \( \hat{K} \) eigenfunctions, which correspond to their eigenvalues \( \lambda_u \) and \( \lambda_T \). In this sense eigenvalues are functions of their eigenfunctions.

Note that the operator \( \hat{K} \) is a Hermitian one when the scalar product is defined by the formula
\[
(a_p, b_p) = \int d^3 p w_p a_p^* b_p.
\] (51)

This is evident from the transformations
\[
(a_p, \hat{K} b_p) = \int d^3 p w_p a_p^* \hat{K} b_p = \{a_p^*, b_p\} = \{b_p, a_p^*\} = \{b_p^*, a_p\} = \left( \int d^3 p w_p b_p^* \hat{K} a_p \right)^* = (b_p, \hat{K} a_p^*),
\] (52)

which take into account relations (44) and (45). This property of the operator \( \hat{K} \) can be used in the analysis of approximate methods for solving equations (39).

5. Method of the truncated expansion in the Sonine polynomials for calculating the theory objects in spatially uniform state

At approximate solution of equations (39) by an expansion in orthogonal polynomials it is expedient to use the Sonine polynomials \( S_q^\alpha(x) \) (\( q = 0, 1, 2, \ldots, \alpha \) is a real number), which are defined by conditions
\[
\int_0^x dx x^\alpha e^{-x} S_q^\alpha(x) S_q^\alpha(x) = \delta_{q_0} \Gamma(q + \alpha + 1) \frac{\Gamma(q + 1)}{q!}, \quad S_0^\alpha(x) = \sum_{0 \leq q \leq q} a_q x^q, \quad a_q = (-1)^q
\] (53)

(see, for example, [9, 11]). This in particular gives
\[
S_0^\alpha(x) = 1, \quad S_1^\alpha(x) = -x + 1 + \alpha, \quad S_2^\alpha(x) = \{x^2 - 2(2 + \alpha)x + (2 + \alpha)(1 + \alpha)\}/2.
\] (54)
The condition of orthonormalization (53) can be written in the form of the average (40) with the Maxwell distribution
\[
\langle e_p^{\alpha-1/2} S_q^{\alpha} (\beta e_p) S_q^{\alpha} (\beta e_p) \rangle = \frac{n}{\beta^{\alpha-1/2}} \frac{2 \Gamma(q+\alpha+1)}{\pi^{1/2} q!} \delta_{qq'} \quad (\beta \equiv T_0^{-1}),
\]
which indicates the expediency of using Sonine polynomials.

Let us find the functions \( A_p \), \( B_p \), and \( C_p \) in the form of expansions
\[
A_p = \sum_{q=0}^{\infty} a_q S_q^{3/2} (\beta e_p), \quad B_p = \sum_{q=0}^{\infty} b_q S_q^{1/2} (\beta e_p), \quad C_p = \sum_{q=0}^{\infty} c_q S_q^{3/2} (\beta e_p),
\]
for which the orthonormalization conditions take the form
\[
\langle p^2 S_q^{3/2} (\beta e_p) S_q^{3/2} (\beta e_p) \rangle = x_q \delta_{qq'}, \quad x_q = \frac{4mn}{\beta} \frac{\Gamma(q+5/2)}{\pi^{3/2} q!},
\]
\[
\langle S_q^{1/2} (\beta e_p) S_q^{1/2} (\beta e_p) \rangle = y_q \delta_{qq'}, \quad y_q = 2n \frac{\Gamma(q+3/2)}{\pi^{1/2} q!}.
\]

Herewith the additional conditions (39) get a simple form
\[
a_0 = \beta, \quad b_0 = 0, \quad b_1 = -\beta, \quad c_0 = 0.
\]

The equations for the coefficients \( a_q \) and \( b_q \) will be found as it follows. We substitute the expansions (56) into equations (39), multiply them by \( p^{3/2} S_p^{3/2} \) and \( S_p^{1/2} \) respectively, take the average value with the distribution function \( w_p \) in both sides and take into account (57). As a result we have
\[
\sum_{q=0}^{\infty} A_{qq'} a_q = \lambda_u a_x x_q, \quad \sum_{q=0}^{\infty} B_{qq'} b_q = \lambda_T b_y y_q, \quad \sum_{q=0}^{\infty} A_{qq'} c_q = v a_x x_q - 3 e n^{1/2} \delta_{q,0},
\]
where the notations
\[
A_{qq'} = \{ p S_q^{3/2} (\beta e_p), p S_q^{3/2} (\beta e_p) \}, \quad B_{qq'} = \{ S_q^{1/2} (\beta e_p), S_q^{1/2} (\beta e_p) \}.
\]
are introduced. The first two equations (59) can also be written in the form
\[
\sum_{q=0}^{\infty} \bar{A}_{qq'} a_q x_q^{1/2} = \lambda_u a_x x_q^{1/2}, \quad \sum_{q=0}^{\infty} \bar{B}_{qq'} b_q y_q^{1/2} = \lambda_T b_y y_q^{1/2}
\]
where
\[
\bar{A}_{qq'} = A_{qq'} (x_q x_q')^{1/2}, \quad \bar{B}_{qq'} = B_{qq'} (y_q y_q')^{1/2}.
\]
Equations (59) in the form (61) are the equations for the eigenvectors \( a_q x_q^{1/2} \) and \( b_q y_q^{1/2} \) and the corresponding eigenvalues \( \lambda_u \) and \( \lambda_T \). Eigenvalues \( \lambda_u \) and \( \lambda_T \) formally are solutions of equations
\[
\det [ \bar{A}_{qq'} - \lambda_u \delta_{qq'} ] = 0, \quad \det [ \bar{B}_{qq'} - \lambda_T \delta_{qq'} ] = 0.
\]
These equations have an infinite set of solutions, from which the minimum ones have to be chosen. Such solutions describe the slowest relaxation process.

Additional conditions (39) give the contribution (58) of the first polynomials to expansion (56). We will say that the solutions of equations (59) are calculated in the \( s \)-polynomial approximation if all the coefficients of the expansion (56) starting from
\((s + 1)\)-th are assumed to be zero. For example, equations (59) and their solutions in the approximation of \(s\) polynomials have the form

\[
\sum_{q=0}^{s} A_{q}a_{q}^{[s]} = \lambda_{u}^{[s]} a_{q}^{[s]} (q = 0, \ldots, s - 1); \quad A_{p}^{[s]} = \sum_{q=0}^{s} a_{q}^{[s]} s_{q}^{[s]} (\beta_{e_{p}}), \quad a_{0}^{[s]} = \beta; \\
\sum_{q=0}^{s} b_{q}^{[s]} b_{q}^{[s]} = \lambda_{r}^{[s]} b_{q}^{[s]} (q = 1, \ldots, s); \quad B_{p}^{[s]} = \sum_{n=1}^{s} b_{n}^{[s]} s_{n}^{[s]} (\beta_{e_{p}}), \quad b_{1}^{[s]} = -\beta.
\]

(64)

Here as unknowns the quantities \(a_{n}^{[s]} (1 \leq n \leq s - 1), \lambda_{u}^{[s]}\) and \(b_{0}^{[s]} (2 \leq n \leq s), \lambda_{r}^{[s]}\) should be considered. Further at quantities for simplicity we will omit an index \(s\), which indicates the number of polynomials in the considered approximation.

In the one-polynomial approximation the equations (59) give

\[
A_{0}a_{0} = \lambda_{u}a_{0}x_{0}, \quad a_{0} = \beta, \quad A_{p} = a_{0} S_{0}^{1/2} (\beta_{e_{p}}); \\
B_{1}b_{1} = \lambda_{r}b_{1}y_{1}, \quad b_{1} = -\beta, \quad B_{p} = b_{1} S_{1}^{1/2} (\beta_{e_{p}}); \\
A_{0}c_{1} = va_{0}x_{0} - 3en^{1/2}, \quad A_{1}c_{1} = va_{1}x_{1}, \quad C_{p} = c_{1} S_{1}^{1/2} (\beta_{e_{p}}),
\]

and therefore

\[
A_{p} = \beta, \quad \lambda_{u} = \frac{\beta}{3\pi n} \{p_{1}, p_{2}\}; \quad B_{p} = -\beta S_{1}^{1/2} (\beta_{e_{p}}), \quad \lambda_{r} = 2\beta^{2} \{e_{p}, e_{p}\}; \\
C_{p} = 0, \quad \nu = \frac{e}{m}.
\]

(65)

The same expressions are given by formulas (47)). The coefficient \(c_{1}\) in (65) should be assumed to be zero, since in the one-polynomial approximation one must take \(a_{1} = 0\).

In the two-polynomial approximation the formulas (59) give the following equations for the quantities \(u_{\lambda_{u}}, a_{1}, \lambda_{r}, b_{2}\) and functions \(A_{p}, B_{p}\)

\[
A_{0}a_{0} + A_{1}a_{1} = \lambda_{u}a_{0}x_{0}, \quad A_{0}a_{0} + A_{1}a_{1} = \lambda_{u}a_{1}x_{1}, \quad A_{p} = a_{0} S_{0}^{1/2} (\beta_{e_{p}}) + a_{1} S_{1}^{1/2} (\beta_{e_{p}}), \quad a_{0} = \beta; \\
B_{1}b_{1} + B_{2}b_{2} = \lambda_{r}b_{1}y_{1}, \quad B_{2}b_{1} + B_{1}b_{2} = \lambda_{r}b_{2}y_{2}, \quad B_{p} = b_{1} S_{1}^{1/2} (\beta_{e_{p}}) + b_{2} S_{2}^{1/2} (\beta_{e_{p}}), \quad b_{1} = -\beta.
\]

(67)

The set of equations (67) gives such an expression for the smallest root \(\lambda_{u}\) of two ones and the expression for the coefficient \(a_{1}\)

\[
\lambda_{u} = \left\{(x_{0} A_{1} + x_{1} A_{0}) - [(x_{0} A_{1} + x_{1} A_{0})^{2} - 4x_{0} x_{1}(A_{1} A_{0} - A_{01} A_{0})]^{1/2}\right\} / 2x_{0} x_{1}, \\
a_{1} = (A_{0} x_{0} - A_{01}) \beta / A_{01}
\]

(68)

Also it gives such an expression for the smallest root \(\lambda_{r}\) of two ones and the expression for the coefficient \(b_{2}\)

\[
\lambda_{r} = \left\{(y_{1} B_{1} + y_{2} B_{2}) - [(y_{1} B_{1} + y_{2} B_{2})^{2} - 4y_{1} y_{2}(B_{1} B_{2} - B_{21} B_{2})]^{1/2}\right\} / 2y_{1} y_{2}, \\
b_{2} = (B_{1} - \lambda_{r}, y_{1}) \beta / B_{21}
\]

(69)

A detailed analysis of the two-polynomial approximation and the convergence of the method of the truncated expansion in the Sonine polynomials for the generalized Lorentz model will be carried out in another paper.
6. Calculation of the relaxation coefficients \( \lambda_u, \lambda_T \) in one-polynomial approximation

In accordance with expression (66), the brackets needed to calculate the relaxation coefficients \( \lambda_u \) and \( \lambda_T \) in the one-polynomial approximation are \( \{ p_i, p_j \} \) and \( \{ e_i, e_j \} \).

Taking into account the definitions of the brackets (44), which include the function \( N_{Dp} \) from (8), one has

\[
\{ a, b \} = 2\pi \alpha \int d^3 p d^3 p' w_P w_{P'} S_{nl} \left( \frac{p}{m} - \frac{p'}{M} \right) \frac{\partial a}{\partial p} \frac{\partial b}{\partial p'}.
\]  

(70)

Let us calculate a more general value, through which the brackets are expressed

\[
I_{nl} = \int d^3 p d^3 p' w_P w_{P'} \left( \frac{p_1 - p_2}{m} \right) f(p_1, p_2).
\]

(71)

In order to do this, let us introduce in the integral the velocities that are used in the two body problem

\[
p_1 = m v_1, \quad p_2 = M v_2; \quad v_i = \frac{m v_1 + M v_2}{m + M}, \quad \nu = v_1 - v_2,
\]

Given the well-known identities of this problem \( d^3 v_1 d^3 v_2 = d^3 v d^3 u \), we have

\[
I_{nl} = (mM)^3 \int d^3 v d^3 u \left( \frac{m n_0}{2\pi T_0 m^{3/2}} \right) \left( \frac{m n_0}{2\pi T_0 M^{3/2}} \right) e^{\frac{(m/M)v^2}{2\nu}} e^{\frac{-2\nu}{2(m/M)\nu}} S_{nl}(v) \times
\]

\[
\times f(m v_1, M v_2).
\]

Another substitution of variables

\[
v_1 = \left( \frac{T_0}{m + M} \right)^{1/2} q_1, \quad \nu = \left( \frac{m + M}{mM} \right)^{1/2} q
\]

finally gives

\[
I_{nl} = \frac{n_0 m^{3/2}}{(1 + \lambda^2)^{1/2} T_0^{1/2}} \int d^3 q_1 d^3 q_2 w_q S_{nl}(q) \overline{T}(q, q_1)
\]

(72)

where

\[
\overline{T}(q, q_1) = f \left( \frac{(mT_0)^{1/2}}{(1 + \mu^2)^{1/2} T_0^{1/2}} (\mu q_1 + q), \frac{(mT_0)^{1/2}}{(1 + \mu^2)^{1/2} T_0^{1/2}} \left( \frac{1}{\mu} q_1 - q \right) \right); \quad w_q = \frac{1}{(2\pi)^3/2} \sin^2 \frac{q^2}{2}
\]

(73)

In the problem of completely ionized plasma the magnitude \( \mu = (m/M)^{1/2} \) is a small parameter, since the mass of an ion \( M \) is much larger than the electron mass \( m \).

Formulas (66) and relations (70)-(73) give the following expressions for the temperature and velocity relaxation coefficients in the one-polynomial approximation

\[
\lambda_u = \frac{2^{5/2} \alpha n_0^{1/2}}{3(1 + \mu^2)^{1/2} m^{1/2} T_0^{1/2}}, \quad \lambda_T = \frac{2^{7/2} \alpha \mu^{1/2} n_0^{1/2}}{3(1 + \mu^2)^{1/2} m^{1/2} T_0^{1/2}} = \frac{2\mu^2}{1 + \mu^2} \lambda_u
\]

(74)

In paper [7] these values were calculated in the main approximation in the parameter \( \mu \).

At the same time our expression for \( \lambda_u \) matches the expression [7], but our expression for \( \lambda_T \) gives the expression of [7] after the replacement \( n_0 \rightarrow n_0(z + 1) \). This result is expected since in [7] the dynamics of ions was more fully taken into account, but the calculations could be carried out only with the additional assumption of \( \mu \) smallness.

7. Conclusions

In the paper based on the Landau’s kinetic equation for a completely ionized plasma a generalized Lorentz model, which considers the system of ions to be equilibrium one, is
developed. As a result, the plasma electron system is described by a Fokker-Planck kinetic equation. In the Bogolyubov reduced description method the basic equations for constructing the hydrodynamics of electrons are developed. They take into account the temperature and macroscopic velocity relaxation processes. The equations of the theory are analyzed at the end of the relaxation processes with restricting ourselves by considering the basic in the gradients approximation, which corresponds to the description of relaxation processes in a spatially homogeneous case. The resulting equations are solved approximately by the method of the truncated expansion in the Sonine polynomials, limiting the consideration by one- and two-polynomial approximation. Finally the coefficients of relaxation of temperature and electron velocity are calculated in the one-polynomial approximation.

References