TAKING INTO ACCOUNT THE INFLUENCE OF CORRELATIONS ON THE SYSTEM DYNAMICS IN THE REDUCED DESCRIPTION METHOD

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A complex of issues related to using the Peletminsky–Yatsenko model for the study of non-equilibrium processes is investigated. The problems are associated with the inability of performing the calculation of necessary averages when system states are described by a new, wider set of reduced description parameters, in particular, fluctuations. A matter system interacting with a non-equilibrium system of bosons is considered. The substance is described by two reduced description parameters η_1, η_2 and calculations with the quasi-equilibrium statistical operator corresponding to these parameters are technically impossible; at the same time, the calculations can be performed with such operator in the case of the only parameter η_1. Overcoming this problem by considering only small values $\delta\eta_2$ of the other reduced description parameter is proposed. An exact definition of $\delta\eta_2$ is given, and the theory of perturbations by powers $\delta\eta_2$ is developed for the calculation of the initial quasi-equilibrium statistical operator with accuracy up to second-order contributions. Since all further averages are calculated only with the named operator including only η_1, the simplified calculation procedure is carried out for it.

On this basis, the theory of non-equilibrium processes in the system of two-level emitters interacting with the boson field is developed. The emitter system is described by excitation degree η_1 and the value of emitter correlations η_2. The system is considered as a system of spins. The mathematically identical Dicke model describing superradiance and Wagner model introduced for the analogous acoustic phenomenon description are studied. The quasi-equilibrium statistical operator calculation is constructed.

Keywords: reduced description method, small parameter, Peltminskii–Yatsenko model, non-equilibrium system of bosons, two-level emitters, Dicke model, Wagner model with tunneling.

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1. Introduction

The Bogolyubov method of the reduced description (RD) of non-equilibrium processes is a powerful tool for studying the dynamics of macroscopic systems [1] (see also [2]). In this work, it is planned to develop it to a certain extent, and therefore it is appropriate to pay attention to its basic ideas.

The Bogolyubov RD method is based on his functional hypothesis [1, 2], according to which the statistical operator of a non-equilibrium system $\rho(t)$ at large times $t >> \tau_0$ depends on time and the initial state of the system $\rho(t=0) \equiv \rho_0$ only through a limited number of parameters $\eta_a(t,\rho_0)$, which are called reduced description parameters (RDPs)

$$\rho(t) \xrightarrow[t>>\tau_0]{} \rho(\eta(t,\rho_0)) \qquad (\mathrm{Sp}\,\rho(\eta)=1, \quad \mathrm{Sp}\,\rho(\eta)\hat{\eta}_a = \eta_a) \tag{1}$$

and defined by the formula

$$\mathrm{Sp}\,\rho(t)\hat{\eta}_a \xrightarrow[t>>\tau_0]{} \eta_a(t,\rho_0)\,. \tag{2}$$

($\hat{\eta}_a$ are RDP operators). *The arrows in formulas* (1), (2) *indicate that their right parts are the asymptotics of the left ones* [2].

The basic statement of the RD method is that *the statistical operator* $\rho(\eta(t,\rho_0))$ *exactly satisfies the Liouville equation* [2]

$$\partial_t \rho(\eta(t,\rho_0)) = \frac{i}{\hbar}[\rho(\eta(t,\rho_0)), \hat{H}] \tag{3}$$

with the Hamilton operator $\hat{H}$, and *the parameters* $\eta_a(t,\rho_0)$ *exactly satisfy the time equation* [2]

$$\partial_t \eta_a(t,\rho_0) = L_a(\eta(t,\rho_0)), \qquad L_a(\eta) \equiv \frac{i}{\hbar}\mathrm{Sp}\,\rho(\eta)[\hat{H},\hat{\eta}_a]. \tag{4}$$

For convenience, *the solutions of equations* (3), (4*) are extended down to* $t = 0$ [2], but they describe the evolution of the system only for $t >> \tau_0$. At the same time, the values of the functions $\eta_a(t,\rho_0)$ at $t = 0$ are called the effective initial conditions.

With account for the function $L_a(\eta)$ expression, the statistical operator $\rho(\eta)$ satisfies a nonlinear differential equation

$$\sum\nolimits_a \frac{\partial \rho(\eta)}{\partial \eta_a} L_a(\eta) = \frac{i}{\hbar}[\rho(\eta), \hat{H}], \tag{5}$$

which can be solved only approximately in some perturbation theory. Particularly important is the case of the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ with main $\hat{H}_0$ and small $\hat{H}_1$ parts (formally $\hat{H} \sim \lambda^0$, $\hat{H}_1 \sim \lambda^1$ where λ is a small parameter).

Further specification of consideration of non-equilibrium states of the system requires selecting the RDPs and their operators $\hat{\eta}_a$. At the same time, the previous developments of the non-equilibrium process theory are taken into account too. The modern trend is to expand the set of RDPs, in particular by taking into account non-equilibrium correlations (fluctuations) of the studied RDPs.

An important approach to the selection of RDPs was proposed by Peletminskii and Yatsenko based on the study of the main Hamiltonian $\hat{H}_0$ symmetries. This led them (see the original paper [3] as well as [2]) to the model (Peletminsky–Yatsenko (PelYats) model) in which the operators of RDPs satisfy the condition

$$[\hat{H}_0, \hat{\eta}_a] = \sum\nolimits_b c_{ab} \hat{\eta}_b \tag{6}$$

where c_{ab} is a numerical matrix. Based on this model, the authors established the functional hypothesis in the main approximation of the perturbation theory [2]

$$e^{-\frac{i}{\hbar}t\hat{H}_0} \rho_0 e^{\frac{i}{\hbar}t\hat{H}_0} \xrightarrow[t>>\tau_0]{} \rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}t}\mathrm{Sp}\rho_0\hat{\eta})). \tag{7}$$

This includes the statistical operator

$$\rho_q(Y) = \exp\left\{\Omega(Y) - \sum\nolimits_a Y_a \hat{\eta}_a\right\}, \qquad \mathrm{Sp}\,\rho_q(Y) \equiv 1, \tag{8}$$

which is called quasi-equilibrium (QESO) because it is close to the equilibrium statistical operator in certain cases. The functions $Z_a(\eta)$ in (7) are determined by the conditions

$$\mathrm{Sp}\rho_q(Z(\eta))\hat{\eta}_a = \eta_a. \tag{9}$$

Bogolyubov showed that *equation* (5) *for the statistical operator* $\rho(\eta)$ *is invariant with respect to time reversal and should be supplemented with a boundary condition that selects the physical direction of time* [1]. As such a condition, the PelYats model authors chose the functional hypothesis (7) in the main approximation of the perturbation theory for the statistical operator with $\rho_0 = \rho(\eta)$, which with account for (1) takes the form

$$e^{-\frac{i}{\hbar}t\hat{H}_0}\rho(\eta)e^{\frac{i}{\hbar}t\hat{H}_0} \xrightarrow[t>>\tau_0]{} \rho_q(Z(e^{\frac{i}{\hbar}ct}\eta) \tag{10}$$

and is recorded in terms of evolution in the physical direction of time. This relation made it possible to obtain a nonlinear integral equation from the nonlinear differential equation (5).

$$\rho(\eta) = \rho_q(Z(\eta)) + \int_{-\infty}^{0} d\tau e^{\frac{i}{\hbar}\hat{H}_0\tau} \mathrm{f}(e^{\frac{i}{\hbar}c\tau}\eta)e^{-\frac{i}{\hbar}\hat{H}_0\tau} \tag{11}$$

where denoted

$$\mathrm{f}(\eta) = \frac{i}{\hbar}\left[\rho(\eta), \hat{H}_1\right] - \sum_a \frac{\partial\rho(\eta)}{\partial\eta_a} M_a(\eta), \qquad M_a(\eta) = \frac{i}{\hbar}\mathrm{Sp}\,\rho(\eta)[\hat{H}_1, \hat{\eta}_a]. \tag{12}$$

In these terms, the right-hand side of the time equation (4) for RDPs with account (4), (6) takes the form

$$L_a(\eta) \equiv \frac{i}{\hbar}\sum_b c_{ab}\eta_b + M_a(\eta) \tag{13}$$

Equation (11) can be solved by an iterative procedure in perturbation theory in $\hat{H}_1$, since its integrand has the first order with respect to $\hat{H}_1$. For the statistical operator $\rho(\eta)$ we have

$$\rho(\eta) = \sum_{n=0}^{\infty}\rho^{(n)}(\eta), \qquad \rho^{(0)}(\eta) = \rho_q(Z(\eta)),$$
$$\rho^{(1)}(\eta) = \int_{-\infty}^{0} d\tau e^{\frac{i}{\hbar}\hat{H}_0\tau}\left\{\frac{i}{\hbar}\left[\rho_q(Z(\eta)), \hat{H}_1\right] - \sum_a \frac{\partial\rho_q(Z(\eta))}{\partial\eta_a} L_a^{(1)}(\eta)\right\}_{\eta\to e^{\frac{i}{\hbar}c\tau}\eta} e^{-\frac{i}{\hbar}\hat{H}_0\tau}. \tag{14}$$

The right-hand side $L_a(\eta)$ of the time equation for RDP is given by the formulas

$$L_a(\eta) = \sum_{n=1}^{\infty} L_a^{(n)}(\eta), \qquad L_a^{(1)}(\eta) = \frac{i}{\hbar}\mathrm{Sp}\rho_q(Z(\eta))\left[\hat{H}_1, \hat{\eta}_a\right],$$
$$L_a^{(2)}(\eta) = \frac{i}{\hbar}\mathrm{Sp}\rho^{(1)}(\eta)\left[\hat{H}_1, \hat{\eta}_a\right]. \tag{15}$$

The creation of the PelYats model was a significant step in the development of the theory of non-equilibrium processes, which made it possible to investigate a number of new problems (see, in particular, [2]).

The structure of this work is as follows. The Introduction discusses the basic concepts of the RD method and the PelYats model in the form necessary for its development in the recent work. In Section 2, a new concept of small RDPs is introduced and, within the framework of the PelYats model, an approach is developed for working with them using the example of matter interacting with a boson field. The task is to overcome the problem of the

PelYats model associated with the inability to perform calculations with QESOs. In Section 3, based on the results of Section 2, the kinetics of a system of two-level emitters interacting with a non-equilibrium system of bosons is developed.

The preliminary results of the work were reported at the MEICS-23 [4] and CM&LTP-23 [5] conferences.

2. Non-equilibrium states with small parameters of the reduced description

The main problem when using the PelYats model to solve concrete problems is the calculation of averages with the QESO $\rho_q(Y)$ defined in (8) and the related functions $\Omega(Y)$, $Z_a(\eta)$ introduced in (8) and (9). For a number of systems, approaches to the calculation of such averages of the type of Wick's rules have been developed. This applies to the situation when the operators in the exponent are quadratic forms in terms of particle creation and annihilation operators (see, for example, [2]) or linear forms in terms of spin operators [6, 7].

In the PelYats model, we consider a system, in which matter interacts with a system of bosons and whose state is described by the average values of three operators $\hat{\eta}_a$: $\hat{\eta}_k \equiv \hat{n}_k, \hat{\eta}_1, \hat{\eta}_2$, $\hat{n}_k \equiv a_k^+ a_k$ where a_k^+, a_k are creation and annihilation operators of bosons in the state with the number k, $\hat{\eta}_1$ and $\hat{\eta}_2$ are operators of physical quantities of matter. For the sake of simplicity, we will assume that all operators commute between themselves and with the main contribution to the Hamiltonian of the system $\hat{H}_0$

$$[\hat{\eta}_a, \hat{\eta}_b] = 0, \qquad [\hat{H}_0, \hat{\eta}_a] = 0. \tag{16}$$

Accordingly to (14) – (16), the statistical operator $\rho(\eta)$ and the right-hand side $M_a(\eta)$ of the equations for RDPs take the form

$$\rho(\eta) = \rho_q + \int_{-\infty}^{0} d\tau \frac{i}{\hbar}[\rho_q, \hat{H}_1(\tau)] + O(\lambda^2),$$
$$M_a(\eta) = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \operatorname{Sp}\rho_q[\hat{H}_1(\tau),[\hat{H}_1,\hat{\eta}_a]] + O(\lambda^3), \quad \hat{H}_1(\tau) \equiv e^{\frac{i}{\hbar}\tau \hat{H}_0}\hat{H}_1 e^{-\frac{i}{\hbar}\tau \hat{H}_0}. \tag{17}$$

Its QESO ρ_q, taking into account all degrees of freedom, has the form

$$\rho_q = \rho_{qB}\rho_{qM}, \quad \rho_{qB} \equiv e^{\Omega_B - \sum_k Z_k \hat{n}_k}, \quad \rho_{qM} \equiv e^{\Omega_M - Z_1\hat{\eta}_1 - Z_2\hat{\eta}_2}, \tag{18}$$

where ρ_{qB}, ρ_{qM} are the contributions of degrees of freedom of bosons and matter. The functions $\Omega_B(Z_{k'})$, $\Omega_M(Z_1, Z_2)$, $Z_k(n_k)$, $Z_1(\eta_1,\eta_2)$, $Z_2(\eta_1,\eta_2)$ in these expressions are defined by formulas

$$\operatorname{Sp}\rho_{qB} = 1, \quad \operatorname{Sp}\rho_{qM} = 1;$$
$$\operatorname{Sp}\rho_{qB}\hat{n}_k = n_k, \quad \operatorname{Sp}\rho_{qM}\hat{\eta}_1 = \eta_1, \quad \operatorname{Sp}\rho_{qM}\hat{\eta}_2 = \eta_2. \tag{19}$$

Averages with the statistical operator ρ_{qB} are calculated using Wick's rules with elementary pairings

$$a_k^+ a_{k'} = n_k \delta_{kk'} , \quad a_k a_{k'}^+ = (1 + n_k)\delta_{kk'} , \quad a_k a_{k'} = 0 \quad a_k^+ a_{k'}^+ = 0 ; \\ n_k = (e^{Z_k} - 1)^{-1} . \tag{20}$$

The functions specified after (18) are given by the formulas

$$Z_k = \ln(1 + n_k^{-1}) , \qquad \Omega_F = \sum\nolimits_k \ln(1 - e^{-Z_k}) \tag{21}$$

(see, for example [2]).

New possibilities of using the PelYats method appear if averages with the statistical operator

$$\rho_q^0 = e^{\Omega^0 - Z_1^0 \hat{\eta}_1} ; \qquad \mathrm{Sp}\,\rho_q^0 = 1 , \quad \mathrm{Sp}\,\rho_q^0 \hat{\eta}_1 = \eta_1 \tag{22}$$

can be calculated and the last two formulas give concrete expressions for the functions

$$\Omega^0(Z_1^0) , \qquad Z_1^0(\eta_1) . \tag{23}$$

This is the case if Wick's type rules are valid for calculations with ρ_q^0. For such situation, we have developed a theory, in which we proposed to limit ourselves to considering a small value (deviation) $\delta\eta_2$ of the parameter η_2 from its average value (along with an arbitrary parameter η_1) for the system with the statistical operator ρ_q^0. The exact definition of deviations is given by the formula

$$\delta\eta_2 \equiv \eta_2 - \mathrm{Sp}\,\rho_q^0 \hat{\eta}_2 = \mathrm{Sp}(\rho_{qM} - \rho_q^0)\hat{\eta}_2 \qquad (\delta\eta_2 \sim \mu , \quad \mu << 1) \tag{24}$$

(μ is a formal small parameter). Taking into account (16) and (17), the QESO takes the form

$$\rho_{qM} = \rho_q^0 + \Delta\rho , \qquad \Delta\rho \equiv \rho_q^0 (e^{F - A\hat{\eta}_1 - B\hat{\eta}_2} - 1) \tag{25}$$

where is noted for simplicity

$$F = \Omega - \Omega^0 , \qquad A = Z_1 - Z_1^0 , \qquad B = Z_2 . \tag{26}$$

Formulas (25) – (26) show that the quantities (26) are small and can be searched in the form of a power series in μ

$$F = F_1 + F_2 + O(\mu^3) , \qquad A = A_1 + A_2 + O(\mu^3) , \qquad B = B_1 + B_2 + O(\mu^3) \tag{27}$$

(the index of a value shows its order in powers of μ). To calculate these values, we will use the equations

$$\mathrm{Sp}(\rho_{qM} - \rho_q^0)\hat{\eta}_1 = 0 , \qquad \mathrm{Sp}(\rho_{qM} - \rho_q^0)\hat{\eta}_2 = \delta\eta_2 , \qquad \mathrm{Sp}(\rho_{qM} - \rho_q^0) = 0 , \tag{28}$$

which follow from formulas (16) – (19). We look for the solution of these equations with respect to F, A, and B in perturbation theory based on the expansion

$$e^{F-A\hat{\eta}_1-B\hat{\eta}_2}-1=(F-A\hat{\eta}_1-B\hat{\eta}_2)+\frac{1}{2}(F-A\hat{\eta}_1-B\hat{\eta}_2)^2+O(\mu^3)=$$
$$=(F_1-A_1\hat{\eta}_1-B_1\hat{\eta}_2)+(F_2-A_2\hat{\eta}_1-B_2\hat{\eta}_2-\hat{C})+O(\mu^3) \qquad (29)$$

where denoted

$$\hat{C}=-\frac{1}{2}(F_1^2+A_1^2\hat{\eta}_1^2+B_1^2\hat{\eta}_2^2-2F_1A_1\hat{\eta}_1-2F_1B_1\hat{\eta}_2+2A_1B_1\hat{\eta}_1\hat{\eta}_2)\,. \qquad (30)$$

Formulas (19), (23), and (24) define the QESO

$$\rho_{qM}=\rho_q^0[1+(F_1-A_1\hat{\eta}_1-B_1\hat{\eta}_2)+(F_2-A_2\hat{\eta}_1-B_2\hat{\eta}_2-\hat{C})+O(\mu^3)] \qquad (31\ )$$

with accuracy up to contributions of the second order in powers of $\delta\eta_2$. Substitution of expression (31) into equation (28) in the first order in μ power gives a system of equations for quantities F_1, A_1, and B_1

$$\begin{aligned}&\mathrm{Sp}\,\rho_q^0(F_1-A_1\hat{\eta}_1-B_1\hat{\eta}_2)\hat{\eta}_1=0\,,\\&\mathrm{Sp}\,\rho_q^0(F_1-A_1\hat{\eta}_1-B_1\hat{\eta}_2)\hat{\eta}_2=\delta\hat{\eta}_2\,,\\&\mathrm{Sp}\,\rho_q^0(F_1-A_1\hat{\eta}_1-B_1\hat{\eta}_2)=0\,,\end{aligned} \qquad (32)$$

and in the second order, it leads to a system of equations for quantities F_2, A_2, and B_2

$$\begin{aligned}&\mathrm{Sp}\,\rho_q^0(F_2-A_2\hat{\eta}_1-B_2\hat{\eta}_2)\hat{\eta}_1=\mathrm{Sp}\,\rho_q^0\hat{C}\hat{\eta}_1\,,\\&\mathrm{Sp}\,\rho_q^0(F_2-A_2\hat{\eta}_1-B_2\hat{\eta}_2)\hat{\eta}_2=\mathrm{Sp}\,\rho_q^0\hat{C}\hat{\eta}_2\,,\\&\mathrm{Sp}\,\rho_q^0(F_2-A_2\hat{\eta}_1-B_2\hat{\eta}_2)=\mathrm{Sp}\,\rho_q^0\hat{C}\,.\end{aligned} \qquad (33)$$

It is appropriate to introduce more compact notations

$$\begin{aligned}&\mathrm{Sp}\,\rho_q^0\hat{\eta}_1=x_1\,,\qquad \mathrm{Sp}\,\rho_q^0\hat{\eta}_2=x_2\,,\\&\mathrm{Sp}\,\rho_q^0\hat{\eta}_1\hat{\eta}_1=x_{11}\,,\qquad \mathrm{Sp}\,\rho_q^0\hat{\eta}_1\hat{\eta}_2=x_{12}\,,\qquad \mathrm{Sp}\,\rho_q^0\hat{\eta}_2\hat{\eta}_2=x_{22}\,.\end{aligned} \qquad (34)$$

Then the system of equations (27) for F_1, A_1, and B_1 takes the form

$$\begin{aligned}&F_1x_1-A_1x_{11}-B_1x_{12}=0\,,\\&F_1x_2-A_1x_{12}-B_1x_{22}=\delta\hat{\eta}_2\,,\\&F_1-A_1x_1-B_1x_2=0\,.\end{aligned} \qquad (35)$$

Excluding the variable F_1, we find a system of equations for quantities A_1 and B_1

$$A_1y_{11}+B_1y_{12}=0\,,\qquad A_1y_{12}+B_1y_{22}=-\delta\eta_2\qquad (\,y_{ab}\equiv\langle\hat{\eta}_a,\hat{\eta}_b\rangle\,) \qquad (36)$$

where correlation functions of quantities a, b with operators $\hat{a}, \hat{b}$ are introduced

$$\langle\hat{a},\hat{b}\rangle\equiv\frac{1}{2}\mathrm{Sp}\,\rho_q^0\{\hat{a},\hat{b}\}-\mathrm{Sp}\,\rho_q^0\hat{a}\,\mathrm{Sp}\,\rho_q^0\hat{b}\,. \qquad (37)$$

As a result, we get the expressions for the coefficients

$$A_1 = \frac{y_{12}}{y_{11}y_{22} - y_{12}^2}\delta\eta_2, \qquad B_1 = \frac{y_{11}}{y_{12}^2 - y_{11}y_{22}}\delta\eta_2, \qquad F_1 = \frac{x_2 y_{11} - x_1 y_{12}}{y_{12}^2 - y_{11}y_{22}}\delta\eta_2, \tag{38}$$

which determine, in accordance with (31), the first-order contribution to the QESO ρ_q. Similarly, it is possible to find the contribution of the second order in $\delta\eta_2$ by calculating the coefficients F_2, A_2, and B_2 from equations (33) rewritten in the form

$$\begin{aligned} &F_2 x_1 - A_2 x_{11} - B_2 x_{12} = \mathrm{Sp}\,\rho_q^0 \hat{C}\hat{\eta}_1, \\ &F_2 x_2 - A_2 x_{12} - B_2 x_{22} = \mathrm{Sp}\,\rho_q^0 \hat{C}\hat{\eta}_2, \\ &F_2 - A_2 x_1 - B_2 x_2 = \mathrm{Sp}\,\rho_q^0 \hat{C}. \end{aligned} \tag{39}$$

Excluding the variable F_2, we find a system of equations for quantities A_2 and B_2

$$A_2 y_{11} + B_2 y_{12} = -\langle \hat{C}, \hat{\eta}_1 \rangle, \qquad A_2 y_{12} + B_2 y_{22} = -\langle \hat{C}, \hat{\eta}_2 \rangle, \tag{40}$$

which is analogous to the system of equations (36). Now, it is easy to write the final expressions for the coefficients F_2, A_2, and B_2.

Note that we have developed a scheme for calculating the QESO ρ_q of the system given in (18) and (31) as a function of the RDPs n_k, η_1, $\delta\eta_2$. The final expressions contain averages with the statistical operator ρ_q^0, which, according to our assumption, can be calculated. Substitution (18), (31) in formulas (17) will give the statistical operator of the system and the right-hand side of the equations for RDP as a function of n_k, η_1, $\delta\eta_2$, i.e. $\rho(n_{k'}, \eta_1, \delta\eta_2)$, $L_a(n_{k'}, \eta_1, \delta\eta_2)$. We calculated these functions with accuracy up to the contributions of the second order in terms of $\delta\eta_2$ powers. Time equations for RDPs, according to (4) and taking into account the definition (24), have the form

$$\begin{aligned} &\partial_t n_k = L_k(n_{k'}, \eta_1, \delta\eta_2), \qquad \partial_t \eta_1 = L_1(n_{k'}, \eta_1, \delta\eta_2), \\ &\partial_t \delta\eta_2 = L_2(n_{k'}, \eta_1, \delta\eta_2) - L_1(n_{k'}, \eta_1, \delta\eta_2)\partial/\partial\eta_1\,\mathrm{Sp}\,\rho_q^0(\eta_1)\hat{\eta}_2. \end{aligned} \tag{41}$$

The developed theory allows numerous applications. Like Bogolyubov's RD method of non-equilibrium processes in general, the discussed PelYats model is based on a careful selection of RDPs and the possibility of using the corresponding perturbation theory. Considerations such as the physical meaning of the approximate correlation gap and approaches to the temporal and spatial localization of the obtained equations (artificial consideration of the weak dependence on the coordinates and time of the objects of the theory) are not taken into account. The developed theory describes the dynamics of matter that interacts with a non-equilibrium system of bosons. The most expected examples are the non-equilibrium electromagnetic field and the phonon field in a solid body, which weakly interact with non-equilibrium matter, which is described by two RDPs η_1, η_2 with simple dynamics in terms of the parameter η_1 and complex dynamics in terms of η_2. At the same time, the complexity of the dynamics is determined by the possibility of studying the quasi-equilibrium state. The developed theory with small values of the parameter η_2 allows its interpretation, in which the parameter η_2 describes small correlations in the system, etc.

3. Dynamics of two-level systems in a bosonic field

An important example of a two-level system is the Dicke model, which was introduced to describe the phenomenon of superradiance. It consists of identical two-level emitters (matter) and a transverse electromagnetic field. The Dicke Hamiltonian of this system in terms of spin operators $\hat{s}_{in}$ and photon Bose operators a_k, a_k^+ has the form [8]

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad \hat{H}_0 \equiv \hbar\omega_0 \sum\nolimits_i \hat{s}_{iz} + \sum\nolimits_k \hbar\omega_k a_k^+ a_k, \quad \hat{H}_1 \equiv -\sum\nolimits_i \hat{E}_n(x_i) d_{in} (\hat{s}_i^+ + \hat{s}_i^-), \qquad (42)$$

where x_i is the radius vector of the *i*-th emitter, $\hbar\omega_0$ is its excitation energy, d_{in} is the dipole moment vector of the corresponding transition, $\hat{E}_n(x)$ is the el**e**ctric field strength operator (standard designations of spin variables are used, in particular $\hat{s}_i^{\pm} = \hat{s}_{ix} \pm i\hat{s}_{iy}$).

The state of the system is described by the average values of the operators $\hat{\eta}_a$:

$$\hat{\eta}_k = \hat{n}_k \equiv a_k^+ a_k, \qquad \hat{\eta}_1 = \hat{s}_z \equiv \sum\nolimits_i \hat{s}_{iz}, \qquad \hat{\eta}_2 = \hat{s}^+ \hat{s}^- \quad (\hat{s}^{\pm} \equiv \sum\nolimits_i \hat{s}_i^{\pm}), \qquad (43)$$

which are denoted by η_a: $\eta_k = n_k$, $\eta_1 = s_z$, η_2 and will be its RDPs. The value n_k describes the non-equilibrium state of the photon system (n_k is the number of photons in the *k*-th state), the value s_z describes the excitation degree of the emitter system, η_2 describes the correlations in the emitter system. It is believed [9] that the description of the system in the Dicke model by the specified RDPs adequately reflects the coherent spontaneous radiation process.

All operators (43) commute with each other and with the main contribution to the Hamiltonian $\hat{H}_0$:

$$[\hat{\eta}_a, \hat{\eta}_b] = 0, \qquad [\hat{H}_0, \hat{\eta}_a] = 0 \qquad (44)$$

since

$$[\hat{s}_{in}, \hat{s}_{i'l}] = i\hbar e_{nlm} \delta_{ii'} \hat{s}_{im}, \quad [\hat{s}_{iz}, \hat{s}_i^{\pm}] = \pm \hat{s}_i^{\pm}, \quad [\hat{s}_{iz}, \hat{s}_i^+ \hat{s}_i^-] = 0, \quad [\hat{s}_{in}, \hat{n}_k] = 0, \quad [\hat{n}_k, \hat{n}_{k'}] = 0. \qquad (45)$$

The relations (44) show that Dicke model study is a special case of the theory from Section 2, built on the PelYats model basis.

Its quasi-equilibrium statistical operator, taking into account all degrees of freedom, has the form

$$\rho_q = \rho_{qB} \rho_{qM}, \quad \rho_{qB} \equiv e^{\Omega_B - \sum_k Z_k \hat{n}_k}, \quad \rho_{qM} \equiv e^{\Omega_M - Z_1 \hat{\eta}_1 - Z_2 \hat{\eta}_2}, \qquad (18)$$

where ρ_{qB}, ρ_{qM}, are the contributions of degrees of freedom of bosons and matter. Accurate calculation of the average values of spin operator functions with a statistical operator ρ_{qM} is impossible because the operator $\hat{\eta}_2$ from (43) is a quadratic form from spin operators (see [6]).

However, calculating averages with a quasi-equilibrium statistical operator

$$\rho_q^0 = e^{\Omega^0 - Z_1 \hat{\eta}_1} = e^{\Omega^0 - Z_1 \sum_i \hat{s}_{iz}} \qquad (46)$$

is possible and this is sufficient to implement the theory from Section 2 for states that are described by a parameter η_1 and a small deviation $\delta\eta_2$ of the parameter η_2 from its average value in the theory without this RDP (see the exact definition in (24)) in the Dicke model. Since in this case η_2 describes the correlations in the emitter system, the theory developed in Section 2 describes this system dynamics when the correlations between emitters are small.

Let us discuss the issue of calculating averages with the statistical operator ρ_q^0. The state space of the *i*-th spin is based on two vectors $|\sigma_i\rangle$

$$\hat{s}_{iz}\,|\sigma_i\rangle = \hbar\sigma_i\,|\sigma_i\rangle\,, \qquad \vec{s}_i^{\,2}\,|\sigma_i\rangle = (\hbar^2 3/4)\,|\sigma_i\rangle \qquad (\sigma_i = \pm 1/2\,). \tag{47}$$

The state space of the *N*-spin system is a direct product of single-spin states. The basis in the space of the system of all spins has the form of a direct product:

$$|n\rangle \equiv |\sigma_1\rangle ... |\sigma_N\rangle\,, \qquad n \equiv \{\sigma_1 ... \sigma_N\}\,; \qquad \sum\nolimits_n ... = \sum_{\sigma_1 ... \sigma_N = \pm 1/2} ...\,, \tag{48}$$

and therefore

$$\rho_q^0\,|n\rangle \equiv w_n\,|n\rangle\,, \qquad w_n = e^{\Omega^0 - Z_1 \sum\limits_{1\le i\le N} \hbar\sigma_i}\,, \qquad \sum\nolimits_n w_n = 1\,, \tag{49}$$

that gives

$$w_n = \prod_{i=1}^{N} w_{\sigma_i}\,, \qquad w_\sigma = (2\,\mathrm{ch}\,Z_1\hbar)^{-1} e^{-Z_1\hbar\sigma}\,. \tag{50}$$

The average value $\bar{a}$ of an arbitrary function of spin operators $\hat{a}$ is given by the formula

$$\bar{a} = \mathrm{Sp}\,\rho_q^0 \hat{a} = \sum\nolimits_n w_n \langle n|\,\hat{a}\,|n\rangle \tag{51}$$

where the matrix element should be calculated using the Pauli matrices. Note that the function of the *i*-th spin variables acts on the basis according to the rule

$$\hat{a}_i\,|\sigma_1\rangle ... |\sigma_i\rangle ... |\sigma_N\rangle = |\sigma_1\rangle ... (\hat{a}_i\,|\sigma_i\rangle) ... |\sigma_N\rangle\,. \tag{52}$$

Thus, a procedure for calculating the averages with the statistical operator ρ_q^0 is obtained. This makes it possible to investigate the dynamics of the Dicke model in the case of small correlations between emitters. To do this, one should first calculate the averages of the RDPs operators given in (34)

$$\begin{gathered} \mathrm{Sp}\,\rho_q^0 \hat{\eta}_1 = x_1\,, \qquad \mathrm{Sp}\,\rho_q^0 \hat{\eta}_2 = x_2\,, \\ \mathrm{Sp}\,\rho_q^0 \hat{\eta}_1\hat{\eta}_1 = x_{11}\,, \quad \mathrm{Sp}\,\rho_q^0 \hat{\eta}_1\hat{\eta}_2 = x_{12}\,, \quad \mathrm{Sp}\,\rho_q^0 \hat{\eta}_2\hat{\eta}_2 = x_{22}\,. \end{gathered} \tag{34}$$

The results of such calculations will be published elsewhere.

Another interesting example of a two-level system that interacts with a boson field is the Wagner model [10] of a solid (see also [11]) with tunneling admixture. In this case, the role of the boson field is played by the phonon system, the peculiarity of which behavior creates a relaxation mechanism in the system modeled by identical two-level emitters following from the Dicke model idea. The Hamiltonian of this system in terms of spin operators $\hat{s}_{in}$ and phonon Bose operators a_k, a_k^+ has the form

$$\hat{H}=\hat{H}_0+\hat{H}_1, \quad \hat{H}_0=\sum\nolimits_k \hbar\omega_k a_k^+ a_k - 2\Delta\sum\nolimits_i s_{iz}\,, \quad \hat{H}_1=\sum\nolimits_{ik} g_i (a_k^+ + a_k)(\hat{s}_i^+ + \hat{s}_i^-)\,, \qquad (52)$$

where the coefficient Δ determines the excitation energy of the system.

This model was introduced to describe the phenomenon of acoustic superradiance. It is believed [11] that this phenomenon is adequately described by the average values of the operators $\hat{\eta}_a$

$$\hat{\eta}_k=\hat{n}_k\equiv a_k^+ a_k\,, \qquad \hat{\eta}_1=\hat{s}_z\equiv\sum\nolimits_i \hat{s}_{iz}\,, \qquad \hat{\eta}_2=\hat{s}^+\hat{s}^- \quad (\hat{s}^\pm\equiv\sum\nolimits_i \hat{s}_i^\pm), \qquad (53)$$

which will be its RDPs. The values n_k describe the non-equilibrium state of the photon system (n_k is the number of photons in the k-th state), the value s_z describes the degree of excitation of the emitter system, the parameter η_2 takes into account correlations in the emitter system. All operators $\hat{\eta}_a$ from (53) commute between themselves and with the main contribution $\hat{H}_0$ to the Hamiltonian. In this approach, the analysis of Wagner model does not differ from the study of the Dicke model.

4. Conclusions

The paper is a continuation of the authors' research of non-equilibrium processes in the framework of the Bogolyubov RD method and its remarkable implementation in the PelYats model (see [5]). The main subject of studies was superradiance phenomenon. This work begins with a discussion of the basic ideas of the RD method and the PelYats model in a form convenient for its development in the work.

In Section 2, it is stated that the main problem of complications when using the PelYats model is the impossibility of calculation performing with the QESO of the system. In this regard, a new idea of a small RDP is put forward. Similar ideas have been discussed in the literature for states close to equilibrium. In our theory, a small RDP is considered to be a deviation of the RDP from its average value in states without this RDP.

On this basis, a system consisting of a boson field and matter that is described by two RDPs is considered. It is supposed that calculations with QESO for η_1, η_2 are impossible, but they are possible with QESO for η_1. In this regard, the description of the system by the parameter η_1 and the small deviation $\delta\eta_2$ of the parameter η_2 is constructed. A perturbation theory is developed, in which the QESO for η_1, η_2 is calculated in $\delta\eta_2$ power series with accuracy up to the second order contributions. It can be directly substituted into the formulas of Section 1 for the statistical operator of the system $\rho(\eta)$ and the right-hand sides $L_a(\eta)$ of the time equations for the RDPs.

Section 3 investigates non-equilibrium two-level systems that interact with a non-equilibrium boson field. Dicke model, which adequately describes the phenomenon of superradiance, and Wagner model, which is able to describe the phenomenon of acoustic superradiatiance, are considered. Mathematically, these models are identical. The parameter η_1, which describes the degree of excitation of the system, and the parameter η_2, which describes the correlations between emitters, are chosen as RDPs for both models. The study of both models is a special case of the one carried out in Section 2. Calculations with QESO for η_1, η_2 are impossible; calculations with QESO for η_1 are not only possible but are developed in detail in our paper. We mean the calculations of averages for spin operators.

Details of the study of the specified models in terms of RDPs η_1, $\delta\eta_2$ will be published in a separate paper.

The completed work opens wide perspectives for further research:

generalization of the method of this work to the case of the general PelYats operator algebra of RDPs;

establishing the connection between the obtained results and the developments in the method of boson variables elimination;

transforming the results related to Dicke model to concrete expressions for time equations for the degree of excitation of the system and the level of correlations between emitters.

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