ON THE QUANTUM MODEL OF A CHARGED BLACK HOLE

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We study the quantum states of the black hole model in the configuration space. To this end, we investigate the properties of the configuration space of the Einstein–Maxwell set of equations in the T-region. We limit ourselves to considering the T-region, where the fields under consideration have a dynamic meaning. Based on the standard action for the of Einstein–Maxwell set of equations, we construct the reduced action for the spherical symmetry case. Using the Hamiltonian constraint, we exclude the non-dynamic degree of freedom from the reduced action, thereby passing to the configuration space. In the new representation of the system, we study the induced dynamical system in the configuration space. It turns out that the induced supermetric is reduced to a quasi-Cartesian form. The laws of charge and mass conservation, which the system contains, together with the Hamilton constraint, completely determine the state of the black hole. They allow one to find the momenta and the action of the system in terms of field variables and conserved quantities, as well as the trajectory of motion in the configuration space. The black hole quantization is reduced to the construction of the quantum states for the system with a fixed mass and charge in a three-dimensional pseudo-Euclidean configuration space The Hamilton constraint is associated with the DeWitt equation, the latter is constructed using the Laplace–Beltrami operator, which is Hermitian with respect to the natural measure. To construct a Hermitian mass operator, it suffices to restrict ourselves to partial derivatives with their corresponding ordering. For the physical states satisfying the DeWitt equation to be also eigenfunctions of the mass operator, the compatibility conditions must be satisfied. In this case, we arrive at the corresponding ansatz. Its substitution into the DeWitt equation leads to a self-consistent solution of the quantum DeWitt equation and equation for the eigenvalue of the mass operator. The constructed model describes the quantum states of a charged black hole in the configuration space with continuous mass and charge spectra.

Keywords: spherically symmetric configuration, minisuperspace, Hamilton operator, mass and charge operators, compatibility condition.

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1. Introduction

One of the most important tools of quantum gravity is the method of the Feynman path integral. However, it can be assumed that the divergence of the two-loop amplitude arising here is related to the general failure of the perturbation theory in general relativity (GR). Therefore, when studying black holes (BH) apparently, we must use a non-perturbative quantization approach. Note that the complexity of some problems can be mitigated with a model approach. One of the popular models is spherically symmetric (SS) configurations of gravitational and electromagnetic fields.

As is well known, the space-time (ST) metric for SS configuration of the electromagnetic and gravitational fields in GR admits the Killing vector. In the R-region, where this vector is timelike, the fields do not have dynamic degrees of freedom. By virtue of this, to study quantization issues, we limited ourselves to considering the T-region, where these fields have a dynamic meaning [1, 2].

2. Classical system states in the configuration space

We write the action of gravitational and the electromagnetic field system in the standard form

$$S = -\frac{1}{16\pi c}\int\left(\frac{c^4}{\kappa}\,{}^{(4)}R + F_{\mu\nu}F^{\mu\nu}\right)\sqrt{-g}\,dx^0 dr\, d\theta\, d\alpha + \text{(boundary terms)} \qquad (1)$$

where ${}^{(4)}R$ is the scalar curvature; $F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}$ is the tensor of the electromagnetic field; $g = \det|g_{\mu\nu}|$. In the T-region with the SS metric

$$ds^2 = \gamma_{ab}dx^a dx^b - R^2(x^0)d\sigma^2 = f(x^0)(dx^0)^2 - h(x^0)(dx^1)^2 - R^2(x^0)d\sigma^2 \quad (2)$$

after dimensional reduction, the action (1) is reduced to the following [1]:

$$S = -\frac{lc^3}{2\kappa}\int\left[\frac{1}{\sqrt{hf}}\left(\frac{\kappa}{c^4}RE^2 - R_{,0}\left(hR\right)_{,0}\right) + \sqrt{hf}\right]dx^0 \quad (3)$$

where $E = F_{01} = A_{1,0} = \varphi_{,0}$. In characteristic variables ξ, R [3]: $f = N^2 R/\xi$, $h = \xi/R$ and the metric (2) takes the form

$$ds^2 = \frac{R}{\xi}\left(Ndx^0\right)^2 - \frac{\xi}{R}\left(dx^1\right)^2 - R^2 d\sigma^2, \quad (4)$$

In this case, the action can be rewritten as follows

$$S = \int L\, dx^0, \qquad L = \frac{1}{2c}\left(N^{-1}I + NU\right), \quad (5)$$

where L is the Lagrangian of the reduced system with kinetic and potential parts

$$I = -\frac{c^4}{\kappa}\dot{\xi}\dot{R} + R^2\dot{\varphi}^2, \qquad U = \frac{c^4}{\kappa} \quad (6)$$

Here it is marked $\dot{\xi} = \xi_{,0}$, $\dot{R} = R_{,0}$, $\dot{\varphi} = \varphi_{,0} = \partial\varphi/\partial x^0$.

From the Lagrange function (5) the constraint and the multiplier N values follow:

$$\frac{\delta L}{\delta N} = \frac{\partial L}{\partial N} = \frac{1}{2c}\left(\frac{I}{N^2} + U\right) = 0\,. \quad (7)$$

$$N = \sqrt{\frac{I}{U}} = \frac{\sqrt{\kappa}}{c^2}\sqrt{I}\,. \quad (8)$$

Using the formula (8), action (5) is converted into action S_T in the configuration space (**CS**):

$$S_T = \int L_{H=0}\, dx^0 = \mu\int\sqrt{I}\, dx^0 = \mu\int d\Omega\,, \quad (9)$$

Herewith

$$dS_T = \mu\sqrt{I}dx^0 = \frac{\mu c^2}{\sqrt{\kappa}}Ndx^0 = \mu d\Omega\,, \quad (10)$$

$$d\Omega^2 = G_{ab}dq^a dq^b = -\frac{c^4}{\kappa}d\xi dR + R^2 d\varphi^2 > 0\,, \quad (11)$$

where $d\Omega^2$ is the **CS** metric in coordinates: $q^a = \{\xi,\ R,\ \varphi\}$, $\mu = cl/\sqrt{\kappa}$.

The determinant $G = \det\|G_{ab}\| = -c^8/4\kappa^2 R^2 < 0$ defines the natural measure in the CP:

$$dV = \sqrt{-G}\, dq^1 dq^2 dq^3 = \frac{c^4}{2\kappa} Rd\xi\, dRd\varphi. \tag{12}$$

Note that with the help of field variable transformations

$$\xi = \eta\left(c\tau - x - \frac{y^2}{R}\right), \qquad \varphi = \frac{1}{\eta}\frac{y}{R}, \qquad R = \eta(c\tau + x), \tag{13}$$

where $\eta = \sqrt{\kappa}/c^2$, the metric (10) is reduced to the Lorentz form

$$d\Omega_0^2 = -c^2 d\tau^2 + dx^2 + dy^2 > 0. \tag{14}$$

From the dynamic system (5), (6) taking into account (8) and (10), we find the momenta

$$P_\xi = -\frac{lc^3}{2\kappa N}\dot{R} = -\frac{lc^5}{2\kappa\sqrt{\kappa}}\frac{dR}{d\Omega}, \qquad P_R = -\frac{lc^3}{2\kappa N}\dot{\xi} = -\frac{lc^5}{2\kappa\sqrt{\kappa}}\frac{d\xi}{d\Omega}$$
$$P_\varphi = \frac{l}{cN}R^2\dot{\varphi} = \frac{cl}{\sqrt{\kappa}}R^2\frac{d\varphi}{d\Omega}. \tag{15}$$

The Legendre transformation

$$NH = P_a\dot{q}^a - L = P_\xi\dot{\xi} + P_R\dot{R} + P_\varphi\dot{\varphi} - L \tag{16}$$

of the dynamical system (5), (6) leads to the Hamiltonian constraint

$$H = \frac{c}{2l}\left\{-\frac{4\kappa}{c^4}P_\xi P_R + \frac{1}{R^2}P_\varphi^2 - \mu^2\right\} = \frac{c}{2l}\left(G^{ab}P_aP_b - \mu^2\right) \square\ 0. \tag{17}$$

Further, we define the following function of variables of electromagnetic field:

$$Q(N,R,\dot{\varphi}) = \frac{c}{l}P_\varphi = \frac{R^2}{N}\dot{\varphi} = \frac{c^2}{\sqrt{\kappa}}R^2\frac{d\varphi}{d\Omega} = const, \tag{18}$$

which persists and is equal to the charge of the configuration within the region of radius R. Adhering to [8], we will call it the charge function of the system.

In addition, there is another conservation law for the SS system of Einstein-Maxwell equations [9]

$$M_{tot} = \frac{c^2}{2\kappa}R\left(1 + \gamma^{ab}R_{,a}R_{,b}\right) + \frac{R^3E^2}{2c^2N^2} = m = const, \tag{19}$$

which is called the mass function. For the T-region with the metric (4), it takes the form

$$M_{tot} = \frac{c^2}{2\kappa}\left[R + \frac{1}{N^2}\left(\xi\dot{R}^2 + \frac{\kappa R^3}{c^4}\dot{\varphi}^2\right)\right] = m = const\cdot \tag{20}$$

Expressing the mass function through momenta P_ξ and P_φ, we obtain

$$M_{tot} = \frac{c^2}{2\kappa}R + \frac{2\kappa}{l^2c^4}\xi P_\xi^2 + \frac{1}{2l^2R}P_\varphi^2. \tag{21}$$

or, taking into account $P_\varphi = lq/c$, we have

$$M_{tot} = \frac{c^2}{2\kappa}R + \frac{2\kappa}{l^2c^4}\xi P_\xi^2 + \frac{q^2}{2c^2R} = m = const\,. \qquad (22)$$

The conservation laws $M_{\rm tot}$ = m and Q = q, together with constraint (17), completely determine the state of the BH. Indeed, they make it possible to express the momenta $\{P_\xi, P_R, P_\varphi\}$ and the action S in terms of field variables $\{\xi, R, \varphi\}$ and conserved quantities q and m.

So from the mass function (22) we find

$$P_\xi = \frac{lc^3}{2\kappa}\sqrt{\frac{R}{\xi}F_T}\,,\;\; F_T = -1 + \frac{2\kappa m}{c^2R} - \frac{\kappa q^2}{c^4R^2}. \qquad (23)$$

Finally, from constraint (17) and relation (22) for the momentum P_R, we find

$$P_R = \frac{lc^3}{2\kappa}\sqrt{\frac{\xi}{RF_T}}\left(\frac{\kappa q^2}{c^4R^2} - 1\right) \qquad (24)$$

To find the action $S(\xi, R, \varphi)$ as a function of field variables and conserved quantities, we write the differential

$$dS = P_\xi d\xi + P_R dR + P_\varphi d\varphi = P_\xi d\xi + P_R dR + (lq/c)\,d\varphi\,\cdot$$

Hence, using the integrability conditions $\partial P_\xi / \partial R = \partial P_R / \partial \xi$ and (23-24), we find $S = 2\xi P_\xi + \frac{lq}{c}\varphi$ or

$$S = S_g + S_q = \frac{lc^3}{\kappa}\sqrt{\xi R F_T} + \frac{l}{c}q\varphi\,\cdot \qquad (25)$$

Further, from the formulas $\partial S/\partial m = la_m$, $\partial S/\partial q = la_q$, where a_m and a_q are constants, we find the trajectories of the system in the CS [8].

3. Quantum states of the system in the configuration states

The quantum states of the field configuration under consideration are determined by the wave function $\Psi(R, \xi, \phi)$ in the CS with the coordinates$\{R, \xi, \phi\}$. We define the momentum operators in the coordinate representation, where instead of partial derivatives the covariant derivatives with respect to the metric (11) are used

$$\hat{P}_\xi = -i\hbar\nabla_\xi, \qquad \hat{P}_R = -i\hbar\nabla_R, \qquad \hat{\mathrm{P}}_\varphi = -i\hbar\nabla_\varphi \qquad (26)$$

The Hamiltonian constraint H = 0 (13) is associated with the quantum analogue ($\hat{H}\Psi = 0$) – the DeWitt equation

$$\hat{H}\Psi = \frac{c}{2l}\left(G^{ab}\hat{P}_a\hat{P}_b - \mu^2\right)\Psi = -\frac{c}{2l}\left(\hbar^2\Delta + \mu^2\right)\Psi \,\square\, 0. \qquad (27)$$

Moreover, the momentum square operator $\hat{P}^2 = G^{ab}\hat{P}_a\hat{P}_b = -\hbar^2\Delta$ is associated with the Laplace–Beltrami operator Δ with respect to the metric (11) in the CS

$$\Delta\Psi = \nabla^a \nabla_a \Psi = \frac{1}{\sqrt{|G|}} \frac{\partial}{\partial q^a} \left(\sqrt{|G|} G^{ab} \frac{\partial \Psi}{\partial q^b} \right). \tag{28}$$

Using the metric (11) and $G = -c^8 / 4\kappa^2 R^2$, we find

$$\Delta\Psi = -\frac{2\kappa}{c^4} \frac{\partial^2 \Psi}{\partial\xi \partial R} - \frac{2\kappa}{c^4} \frac{1}{R} \frac{\partial}{\partial R} R \frac{\partial \Psi}{\partial R} + \frac{1}{R^2} \frac{\partial^2 \Psi}{\partial \varphi^2}. \tag{29}$$

The introduced operator Δ is Hermitian with respect to the natural measure (12) of the CS.

To construct the Hermitian total mass operator, in CS with the volume element (12), it is sufficient to use the following ordering of operators: $\xi P_\xi^2 \Rightarrow \hat{P}_\xi \xi \hat{P}$. It turns out here that the momenta entering into the functions of charge (18) and mass (21) can be associated with partial derivatives: $\hat{P}_\xi = -i\hbar \partial/\partial\xi$, $\hat{P}_\varphi = -i\hbar \partial/\partial\varphi$. Thus, the charge (18) and the total mass (21) correspond to operators

$$\hat{Q} = \frac{c}{l} P_\varphi = -i \frac{c\hbar}{l} \frac{\partial}{\partial\varphi}, \;\; \hat{M} = \frac{1}{2l^2} \left(\frac{l^2 c^2}{\kappa} R - \frac{4\kappa\hbar^2}{c^4} \frac{\partial}{\partial\xi} \xi \frac{\partial}{\partial\xi} - \frac{\hbar^2}{R} \frac{\partial^2}{\partial\varphi^2} \right) \tag{30}$$

The introduced operators satisfy the commutation relations

$$\left[\hat{H}, \hat{M} \right] = -\frac{2\kappa\hbar^2}{l^2 c^4} \frac{\partial}{\partial\xi} \hat{H} \sim 0, \quad \left[\hat{H}, \hat{Q} \right] = 0, \quad \left[\hat{M}, \hat{Q} \right] = 0 \tag{31}$$

Further, we construct states with the fixed mass and charge, i.e., states corresponding to the eigenfunctions and eigenvalues of the mass and charge operators

$$\hat{Q}\psi_q = q\psi_q, \qquad \hat{M}\psi_m = m\psi_m. \tag{32}$$

Writing the first equation expanded form, we obtain

$$-\frac{ic\hbar}{l} \frac{\partial}{\partial\varphi} \psi_q = q\psi_q \quad \Rightarrow \quad \psi_q = A e^{i(ql/c\hbar)\varphi}$$

Therefore, the general wave functions of the DeWitt and charge operators, as well as the operators of the mass and charge, can be represented in the form

$$\begin{aligned} \Psi &= \psi(\xi, \mathrm{R}) \psi_q = \psi(\xi, \mathrm{R}) e^{i(ql/c\hbar)\varphi}, \\ \Psi_m &= \psi_m(\xi, \mathrm{R}) \psi_q = \psi_m(\xi, \mathrm{R}) e^{i(ql/c\hbar)\varphi}. \end{aligned} \tag{33}$$

where the functions ψ and ψ_m satisfy the equations

$$\left(\frac{2\kappa\hbar^2}{c^4} \frac{\partial^2}{\partial\xi \partial R} + \frac{2\kappa\hbar^2}{c^4 R} \frac{\partial}{\partial R} R \frac{\partial}{\partial\xi} + \frac{l^2 q^2}{c^2 R^2} - \frac{c^2 l^2}{\kappa} \right) \psi = 0, \tag{34}$$

$$\left(\frac{l^2 c^2}{\kappa} R - \frac{4\kappa\hbar^2}{c^4} \frac{\partial}{\partial\xi} \xi \frac{\partial}{\partial\xi} + \frac{l^2 q^2}{c^2 R^2} \right) \psi_m = 2l^2 m \psi_m. \tag{35}$$

In the Planckian and corresponding dimensionless quantities

$$m_{pl}^2=\frac{c\hbar}{\kappa},\quad l_{pl}^2=\frac{\hbar\kappa}{c^3},\quad q_{pl}=m_{pl}\sqrt{\kappa}=\sqrt{c\hbar},$$

$$\nu=\frac{m}{m_{pl}},\quad \sigma=\frac{q}{q_{pl}},\quad x=\frac{\xi}{l_{pl}},\quad y=\frac{R}{l_{pl}},\quad \chi=\frac{l}{l_{pl}}$$

the system of differential equations (34) and (35) can be rewritten as follows

$$\frac{\partial^2\psi}{\partial x\partial y}=-\frac{1}{2y}\frac{\partial\psi}{\partial x}+\frac{\chi^2}{4}\left(1-\frac{\sigma^2}{y^2}\right)\psi, \tag{36}$$

$$x\frac{\partial^2\psi_m}{\partial x^2}=-\frac{\partial\psi_m}{\partial x}+\frac{\chi^2}{4}\left(y-2\nu+\frac{\sigma^2}{y}\right)\psi_m. \tag{37}$$

According to (31), the mass operator *M* weakly commutes with the Hamiltonian *H*. In order for the physical states ψ satisfying the DeWitt equation (34) to be also eigenfunctions of the mass operator (35), the set of equations (36), (37) must have a common solution $\psi=\psi_m$. The compatibility condition of this set

$$\frac{\partial}{\partial x}\frac{\partial^2\psi}{\partial x\partial y}-\frac{\partial}{\partial y}\frac{\partial^2\psi}{\partial x^2}=0$$

leads to the equation

$$\left(1-\frac{\sigma^2}{y^2}\right)\frac{\partial\psi}{\partial x}-\frac{1}{x}\left(y-2\nu+\frac{\sigma^2}{y}\right)\frac{\partial\psi}{\partial y}=\frac{1}{2yx}\left(y-2\nu+\frac{\sigma^2}{y}\right)\psi \tag{38}$$

After the replacement $\psi=e^z$, we arrive to the linear inhomogeneous partial differential equation of the first order

$$\left(1-\frac{\sigma^2}{y^2}\right)\frac{\partial z}{\partial x}-\frac{1}{x}\left(y-2\nu+\frac{\sigma^2}{y}\right)\frac{\partial z}{\partial y}=\frac{1}{2yx}\left(y-2\nu+\frac{\sigma^2}{y}\right) \tag{39}$$

From its characteristic equation

$$\frac{dx}{1-\sigma^2/y^2}=-\frac{xdy}{y-2\nu+\sigma^2/y}=\frac{2xydz}{y-2\nu+\sigma^2/y} \tag{40}$$

two integrals follow:

$$e^z\sqrt{y}=C_1,\qquad x\left(y-2\nu+\frac{\sigma^2}{y}\right)=C_2 \tag{41}$$

where C_1 and C_2 are arbitrary constants. The common integral can be represented as

$$e^z=\frac{1}{\sqrt{y}}F\left(x\left(y-2\nu+\frac{\sigma^2}{y}\right)\right)$$

Here *F* is an arbitrary function of its argument. Substituting this result, for example, into the equation (36), we obtain

$$F''+xZF'-\frac{\chi^2}{4}F=0$$

where F' – is the derivative of the function F with respect to the argument. The solution of this equation, regular on the horizon $F_T = 0$, has the form

$$\psi(x,y)=\frac{C}{\sqrt{y}}J_0\left(\chi\sqrt{x\left(2\nu-y-\frac{\sigma^2}{y}\right)}\right). \tag{42}$$

where J_0 is Bessel functions of the first kind of zero order.

Returning to the original variables and taking into account $\psi=e^z$ and (33), we obtain the following, regular on the horizon solution

$$\Psi(\xi,R,\varphi)=\psi e^{iql\varphi/c\hbar}=C\sqrt{\frac{l_{pl}}{R}}J_0\left(\chi\sqrt{\frac{l}{l_{pl}^2}(\xi RF_T)}\right)e^{iql\varphi/c\hbar} \tag{43}$$

where the function F_T is defined in (23). As a result, we get a model of a charged BH with a continuous spectrum of mass m and charge q.

4. Conclusions

The considered field configuration allows two integrals of motion: total mass and charge, so the dynamical system turns out to be completely integrable. Together with the constraint, they completely determine the momenta and the action, as the functions of field variables and conserved quantities in the CS. Moreover, from the action you can find the trajectory of the system motion in the CS. This defines the classical picture in CS. We also carried out the quantization of a black hole by constructing the quantum states of the system with the certain mass and charge in a three-dimensional pseudo-Euclidean CS. For that, we constructed the DeWitt equation with the Laplace-Beltrami operator, which is Hermitian with respect to the natural measure in the CS. For building the Hermitian mass operator, we can restrict ourselves to partial derivatives, but with their corresponding ordering. The compatibility condition for the DeWitt equation and the equation for the eigenfunctions of the mass operator is solved by the method of characteristics and leads to the corresponding ansatz. Using this ansatz, we have obtained a self-consistent solution of the DeWitt equation and the equation for the eigenvalue of the mass operator. As a result, we obtained the charged BH model, whose quantum states in the CS are described by a wave function with continuous mass and charge spectra. Note that the formulas obtained here are consistent with the results of [10] obtained by a different method.

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