FLUCTUATION KINETICS IN SYSTEM IN THE PRESENCE OF RANDOM EXTERNAL FIELD AND A GENERALIZATION OF THE FLUCTUATION-DISSIPATION THEOREM

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The evolution equation of general form for the parameters of the reduced description (RDP) of system in the presence of a random external field is used to describe the dynamics of the system. The field causes fluctuations of RDP. With taking into account the fluctuations, the system is described along with the average value of the RDP itself by the average values of all RDP products (fluctuations) or their correlation functions. For the generating function of these quantities, a closed time equation of the fluctuation kinetics is derived. The general initial form of the time equation for RDP allows investigating kinetic and hydrodynamic states in a unique way without specifying the spatial dependence of quantities. Compared with the known previous works, this greatly simplified the study. The closed time equation for the generating function (the equation of the fluctuation kinetics) is derived using the generalized Furutzu–Novikov theorem, the proof of which is simplified in the paper. The external field is considered as a Gaussian stationary process with a correlation time much shorter than the characteristic time of system evolution. On this basis, a small parameter is introduced, and the corresponding perturbation theory is built. Cases of the field which is introduced through RDP and directly (additive field) are considered. The definition of a generalized nonlinear fluctuation-dissipation theorem is proposed. To illustrate the developed fluctuation kinetics, the approximations of binary correlations are considered, in which more complex correlations are neglected, as well as the states around equilibrium. Fluctuation hydrodynamics, which is compared with the Landau–Lifshitz theory, is considered as an application.

Keywords: reduced description, random external field, fluctuation-dissipation theorem, correlations, generating function, closed equation, fluctuation kinetics.

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1. Introduction

The reduced description of nonequilibrium systems taking into account fluctuations as additional parameters of the reduced description (RDP) is the leading direction of modern research in this field. It is a question of using along with average values $\mathrm{f}_a(t)=\overline{\hat{\eta}_a(t)}$ of some microscopic quantities $\hat{\eta}_a(t)$ all their fluctuations $\mathrm{f}_{a_1\ldots a_n}(t)=\overline{\hat{\eta}_{a_1}(t)\ldots\hat{\eta}_{a_n}(t)}$ $(2\le n<\infty)$ (instead of which it is convenient to use corresponding correlations). A number of such studies have been performed in terms of multiparticle distribution functions $\mathrm{f}_1(x_1,t)$, $\mathrm{f}_n(x_1,\ldots,x_n,t)$ $(2\le n<\infty)$. At the same time, the use of distribution functions smoothed on a microscopic scale proved to be fruitful. The results of such studies are summarized in [1, 2], in which the source of fluctuations was the uncertainty of the initial state of the system. Over time, the idea of using the solutions of time equations for the RDP of the system as microscopic values $\hat{\eta}_a(t)$ has emerged. Uncertainty of the initial state of the system or the presence of a random external field was considered as a source of fluctuations. The results of such studies are summarized in [3], in which a random external field was considered as the source of fluctuations.

Current work is also devoted to the kinetics of the system considering fluctuations. The starting point is the time equation for a reduced description of the nonequilibrium state of the system by some parameters $\eta_a(t)$ in the presence of a random external field $h_i(t)$. These can be kinetic equations, in which the system is described by a one-particle distribution function $\mathrm{f}_p(x,t)$, hydrodynamic equations, in which the system is described by the densities of additive integrals of motion $\zeta_\mu(x,t)$, and various generalizations. A general approach is being developed, which, in contrast to [3], does not require a separate consideration of kinetic and hydrodynamic states and does not contain a cumbersome consideration of the spatial inhomogeneity of the system state in the formalism. The obtained equations allow concretization for kinetic and hydrodynamic states by substitutions of type $\eta_a(t)\to \mathrm{f}_p(x,t)$, $\eta_a(t)\to\zeta_\mu(x,t)$. On this basis, a nonlinear generalization of the fluctuation-dissipation theorem is proposed (previously, such an approach was proposed in our work [4]). The relevance of such studies of fluctuation kinetics is confirmed by reviews [5-7].

The work is structured as follows. In Section 2, a general theory of a reduced description of nonequilibrium systems in the presence of a random external field is constructed. In Section 3, a generalized nonlinear fluctuation-dissipation theorem is formulated and its application to the fluctuation hydrodynamics is discussed.

2. General equations of the reduced description of system in the presence of random external field

The equation for the parameters of the reduced description $\eta_a(t,h)$ in the presence of an external field $h_i(t)$ has the form

$$\partial_t\eta_a(t,h)=L_a(\eta(t,h))+\sum\nolimits_i s_{ai}(\eta(t,h))h_i(t)\,, \tag{1}$$

where $L_a(\eta)$, $s_{ai}(\eta)$ are some functions ($\partial_t\equiv\partial/\partial t$). RDP are considered as functionals of the external field $h_i(t)$. In the case of a random field $\hat{h}_i(t)$, RDP become random variables $\eta_a(t,\hat{h})\equiv\hat{\eta}_a(t)$ (random variables are denoted by a hat). In definition (1) a typical form of the kinetic equation

$$\partial_t \mathrm{f}_p(x,t)=-\frac{p_n}{m}\frac{\partial \mathrm{f}_p(x,t)}{\partial x_n}-F_n(x,t)\frac{\partial \mathrm{f}_p(x,t)}{\partial p_n}+I_p(x,\mathrm{f}(t))\,,$$

and set of hydrodynamic equations

$$\partial_t\rho(x,t)=-\frac{\partial\pi_l(x,t)}{\partial x_l}\,,\qquad \partial_t\pi_n(x,t)=-\frac{\partial t_{nl}(x,\zeta(t))}{\partial x_l}+\frac{1}{m}F_l(x,t)\rho(x,t)\,,$$

$$\partial_t\varepsilon(x,t)=-\frac{\partial q_n(x,\zeta(t))}{\partial x_l}+\frac{1}{m}F_l(x,t)\pi_l(x,t)$$

are included, where $\mathrm{f}_p(x,t)$ is one particle distribution function, $\zeta_\mu(x,t):\rho(x,t),\pi_n(x,t),$ $\varepsilon(x,t)$ are the densities of mass, momentum and energy of the system, $F_l(x,t)$ is the force acting on the particle at a point x.

Properties of quantities $\hat{\eta}_a(t)$ are characterized by their average value $\mathrm{f}_a(t)$ and all fluctuations (average values of all their products)

$$\mathrm{f}_{a_1}(t)=\overline{\hat{\eta}_a(t)}\,,\qquad \mathrm{f}_{a_1\ldots a_n}(t)=\overline{\hat{\eta}_{a_1}(t)\ldots\hat{\eta}_{a_n}(t)}\quad(n\geq 2\,). \tag{2}$$

Average value is taken over all realizations of the field $\hat{h}_i(t)$. Instead of fluctuations it is convenient to use corresponding correlations (correlation functions) $g_{a_1\ldots a_n}(t)$

$$\mathrm{f}_{a_1 a_2}(t) \equiv \mathrm{f}_{a_1}(t)\mathrm{f}_{a_2}(t) + g_{a_1 a_2}(t),$$
$$\mathrm{f}_{a_1 a_2 a_3}(t) \equiv \mathrm{f}_{a_1}(t)\mathrm{f}_{a_2}(t)\mathrm{f}_{a_3}(t) + g_{a_1 a_2}(t)\mathrm{f}_{a_3}(t) + g_{a_1 a_2}(t)\mathrm{f}_{a_3}(t) + g_{a_2 a_3}(t)\mathrm{f}_{a_1}(t) + g_{a_1 a_2 a_3}(t) \tag{3}$$

and so on. Generating functions for these quantities are given by formulas

$$\mathbf{F}(\mathrm{f}(t),u) = 1 + \sum_{1\le n<\infty} \frac{1}{n!} \sum_{a_1 \dots a_n} u_{a_1}\dots u_{a_n} \mathrm{f}_{a_1\dots a_n}(t) = \overline{e^{\sum_a u_a \hat{\eta}_a(t)}},$$
$$\mathbf{G}(g(t),u) = \sum_{2\le n<\infty} \frac{1}{n!} \sum_{a_1 \dots a_n} u_{a_1}\dots u_{a_n} g_{a_1\dots a_n}(t) \tag{4}$$

and related by the formula

$$\mathbf{F}(\mathrm{f}(t),u) = e^{\sum_a \mathrm{f}_a(t) u_a + \mathbf{G}(g(t),u)} \tag{5}$$

(see, for example, [8]).

Let obtain an evolution equation for the functional $\mathbf{F}(\mathrm{f}(t),u)$. Taking into account (1) and (4), we have

$$\partial_t \mathbf{F}(\mathrm{f}(t),u) = \overline{e^{\sum_a u_a \hat{\eta}_a(t)} \sum_a u_a [L_a(\hat{\eta}(t)) + \sum_i s_{ai}(\hat{\eta}(t)\hat{h}_i(t)]}. \tag{6}$$

For further conversion of this expression, the identities

$$\overline{e^{\sum_a \hat{\eta}_a(t) u_a} \varphi(\hat{\eta}(t))} = \overline{e^{\sum_a \hat{\eta}_a(t)[u_a + \frac{\partial}{\partial \eta_a}]}} \varphi(\eta)\Big|_{\eta=0} = \mathbf{F}(\mathrm{f}(t), u + \frac{\partial}{\partial \eta}) \varphi(\eta)\Big|_{\eta=0} =$$
$$= e^{\sum_a \mathrm{f}_a(t)[u_a + \frac{\partial}{\partial \eta_a}] + \mathbf{G}(g(t), u + \frac{\partial}{\partial \eta})} \varphi(\eta)\Big|_{\eta=0} = e^{\sum_a \mathrm{f}_a(t) u_a} e^{\mathbf{G}(g(t), u + \frac{\partial}{\partial \eta})} \varphi(\eta + \mathrm{f(t)})\Big|_{\eta=0} =$$
$$= e^{\sum_a \mathrm{f}_a u_a} e^{\mathbf{G}(g(t), u + \frac{\partial}{\partial \eta})} \varphi(\eta + \mathrm{f})\Big|_{\eta=0, \mathrm{f}=\mathrm{f(t)}} = e^{\sum_a \mathrm{f}_a u_a} e^{\mathbf{G}(g(t), u + \frac{\partial}{\partial \mathrm{f}})} \varphi(\eta + \mathrm{f})\Big|_{\eta=0, \mathrm{f}=\mathrm{f(t)}} = \tag{7}$$
$$= e^{\sum_a \mathrm{f}_a u_a} e^{\mathbf{G}(g(t), u + \frac{\partial}{\partial \mathrm{f}})} \varphi(\mathrm{f})\Big|_{\mathrm{f}=\mathrm{f(t)}},$$
$$\overline{e^{\sum_a \hat{\eta}_a(t) u_a} \varphi(\hat{\eta}(t)) \psi(\hat{\eta}(t))} = \varphi\left(\frac{\partial}{\partial u}\right) \overline{e^{\sum_a \hat{\eta}_a(t) u_a} \psi(\hat{\eta}(t))},$$

which take into account (4) and (5) ($\varphi(\eta)$, $\psi(\eta)$ are arbitrary functions), are needed. Then equation (6) can now be written as

$$\partial_t \mathbf{F}(\mathrm{f}(t),u) = e^{\sum_a \mathrm{f}_a u_a} e^{\mathbf{G}(g(t), u + \frac{\partial}{\partial \mathrm{f}})} \sum_a u_a L_a(\mathrm{f})\Big|_{\mathrm{f}=\mathrm{f(t)}} + \sum_i A_i(u) \Phi_i(t,u) \tag{8}$$

where it is denoted

$$A_i(u) \equiv \sum\nolimits_a u_a s_{ai}\left(\frac{\partial}{\partial u}\right), \qquad \Phi_i(t,u) = \overline{\hat{h}_i e^{\sum_a \hat{\eta}_a(t) u_a}} . \tag{9}$$

In order to calculate the function $\Phi_i(t,u)$, let introduce the average value of the external field $\hat{h}_i(t)$, its fluctuations $m_{i_1\ldots i_n}(t_1 \ldots t_n)$ and correlations $n_{i_1\ldots i_n}(t_1 \ldots t_n)$

$$m_i(t) = \overline{\hat{h}_i(t)}, \qquad m_{i_1\ldots i_n}(t_1 \ldots t_n) = \overline{\hat{h}_{i_1}(t_1) \ldots \hat{h}_{i_n}(t_n)} \quad (n \geq 2). \tag{10}$$

Generating functions for these quantities are given by formulas

$$\mathbf{M}(m,\upsilon) = 1 + \sum_{1\leq n<\infty} \frac{1}{n!} \sum_{i_1\ldots i_n} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n \upsilon_{i_1}(t_1) \ldots \upsilon_{i_n}(t_n) m_{i_1\ldots i_n}(t_1 \ldots t_n) = \overline{e^{\sum_i \int_{-\infty}^{\infty} \hat{h}_i(t)\upsilon_i(t)}} ,$$

$$\mathbf{N}(n,\upsilon) = \sum_{2\leq n<\infty} \frac{1}{n!} \sum_{i_1\ldots i_n} \int_{-\infty}^{\infty} dt_1 \ldots \int_{-\infty}^{\infty} dt_n \upsilon_{i_1}(t_1) \ldots \upsilon_{i_n}(t_n) n_{i_1\ldots i_n}(t_1 \ldots t_n) \tag{11}$$

and are related by a relation like (5)

$$\mathbf{M}(m,\upsilon) = e^{\sum_i \int_{-\infty}^{\infty} dt\, m_i(t)\upsilon_i(t) + \mathbf{N}(n,\upsilon)} \tag{12}$$

Let note further that the relation

$$\overline{\hat{h}_i(t) a(\hat{h})} = m_i(t)\overline{a(\hat{h})} + \overline{a_i(\hat{h})}, \qquad a_i(t,h) \equiv \left.\frac{\delta \mathbf{N}(n,\upsilon)}{\delta \upsilon_i(t)}\right|_{\upsilon = \frac{\delta}{\delta h}} a(h), \tag{13}$$

where $a(h)$ is a functional of $h_i(t)$, is true. It can be called the generalized Furutzu–Novikov formula. Let us prove it by simplifying the proof given in [1]. Consistently it is obtained:

$$\overline{\hat{h}_i(t) a(\hat{h})} \underset{h,\upsilon=0}{=} \overline{e^{\sum_i \int_{-\infty}^{\infty} dt\, \hat{h}_i(t)\left(\upsilon_i(t) + \frac{\delta}{\delta h_i(t)}\right)} \hat{h}_i(t) a(t)} \underset{h,\upsilon=0}{=} \overline{\frac{\delta}{\delta \upsilon_i(t)} e^{\sum_i \int_{-\infty}^{\infty} dt'\, \hat{h}_i\left(\upsilon_i(t') + \frac{\delta}{\delta h_i(t')}\right)} a(t)} =$$

$$= \frac{\delta}{\delta \upsilon_i(t)} \mathbf{M}(m, \upsilon + \frac{\delta}{\delta h}) a(t) = \frac{\delta}{\delta \upsilon_i(t)} e^{\sum_i \int_{-\infty}^{\infty} dt'\, m_i(t')\left(\upsilon_i(t') + \frac{\delta}{\delta h_i(t')}\right) + \mathbf{N}(n, \upsilon + \frac{\delta}{\delta h})} a(h) =$$

$$= e^{\sum_i \int_{-\infty}^{\infty} dt'\, m_i(t')\left(\upsilon_i(t') + \frac{\delta}{\delta h_i(t')}\right) + \mathbf{N}(n, \upsilon + \frac{\delta}{\delta h})} \left(m_i(t) + \frac{\delta \mathbf{N}(n,\upsilon)}{\delta \upsilon_i(t)} \right)_{\upsilon \to \upsilon + \frac{\delta}{\delta h}} a(h) =$$

$$= \mathbf{M}(m, \upsilon + \frac{\delta}{\delta h}) \left(m_i(t) + \frac{\delta \mathbf{N}(n,\upsilon)}{\delta \upsilon_i(t)} \right)_{\upsilon \to \upsilon + \frac{\delta}{\delta h}} a(h) \underset{h,\upsilon=0}{=}$$

$$= \mathbf{M}(m, \frac{\delta}{\delta h}) \left(m_i(t) + \frac{\delta \mathbf{N}(n,\upsilon)}{\delta \upsilon_i(t)} \right)_{\upsilon \to \frac{\delta}{\delta h}} a(h) = \overline{\left(m_i(t) + \frac{\delta \mathbf{N}(n,\upsilon)}{\delta \upsilon_i(t)} \right)_{\upsilon \to \upsilon + \frac{\delta}{\delta h}} a(h)\Big|_{h \to \hat{h}}} ,$$

which, taking into account formulas of type (7), proves (13) (here $\underset{h,\upsilon=0}{=}$ denotes equality, that is true under condition $h,\upsilon=0$).

Further it is assumed that the external field $\hat{h}_i(t)$ is a stationary Gaussian process for which

$$\mathbf{N}(n,\upsilon)=\frac{1}{2}\sum_{ii'}\int_{-\infty}^{\infty}dt\int_{-\infty}^{\infty}dt'\upsilon_i(t)\upsilon_{i'}(t')n_{ii'}(t,t'),$$
$$m_i(t)=m_i, \qquad n_{ii'}(t,t')=n_{ii'}(t-t'), \qquad n_{ii'}(t)=n_{i'i}(t). \tag{14}$$

In this case, formula (13) gives

$$a_i(t,h)\equiv\sum\nolimits_{i'}\int_{-\infty}^{\infty}dt'n_{ii'}(t-t')\frac{\delta a(h)}{\delta h_{i'}(t')} \tag{15}$$

and allows, taking into account (9) and (15), to represent the function $\Phi_i(t,u)$ from (9) in the form

$$\Phi_i(t,u)=m_i\mathbf{F}(\mathrm{f}(t),u)+\sum\nolimits_{a,i'}\int_{-\infty}^{t}dt'n_{ii'}(t-t')u_a\overline{\chi_{ai'}(t,t',\hat{h})e^{\sum_a\hat{\eta}_a(t)u_a}} \tag{16}$$

where the field susceptibility is introduced, and causality considerations are taken into account

$$\chi_{ai}(t,t',h)=\frac{\delta\eta_a(t,h)}{\delta h_i(t')}, \qquad \chi_{ai}(t,t',\hat{h})\underset{t<t'}{=}0. \tag{17}$$

Let introduce the correlation time τ_0 of the random field $\hat{h}_i(t)$

$$n_{ii'}(t-t')\underset{t-t'>>\tau_0}{=}0 \tag{18}$$

and the characteristic time T of the evolution of RDP $\eta_a(t,h)$, assuming that $T>>\tau_0$. In this case $\chi_{ai}(t,t',\hat{h})$ in (16) can be expanded in powers of $t'-t$ near $t'=t-0$

$$\chi_{ai}(t,t',h)=\chi_{ai0}(t,h)+\chi_{ai1}(t,h)(t'-t)+\frac{1}{2}\chi_{ai2}(t,h)(t'-t)^2+\ldots,$$
$$\chi_{ais}(t,h)\equiv\left.\frac{\partial^s\chi_{ai}(t,t',h)}{\partial t'^s}\right|_{t'=t-0}\sim\frac{1}{T^s}. \tag{19}$$

Given this expansion in (16), we have

$$\Phi_i(t,u)=m_i\mathbf{F}(\mathrm{f}(t),u)+$$
$$+\sum\nolimits_{a,i'}u_a\overline{[n_{ii'0}\chi_{ai'0}(t,\hat{h})-n_{ii'1}\chi_{ai'1}(t,\hat{h})+n_{ii'2}\chi_{ai'2}(t,\hat{h})+O(\lambda^3)]e^{\sum_{a'}\hat{\eta}_{a'}(t)u_{a'}}}, \tag{20}$$
$$n_{ii's}\equiv\int_0^{\infty}d\tau n_{ii'}(\tau)\tau^s, \qquad n_{ii's}\sim\tau_0^s.$$

Taking into account (19), we see that $\Phi_i(t,u)$ has the form of an expansion in powers of $\lambda \equiv \tau_0 / T$. It is convenient to calculate the functions $\chi_{ais}(t,h)$ by writing equation (1) in an integral form

$$\eta_a(t,h) = \eta_a(0,h) + \int_0^t dt''[L_a(\eta(t'',h)) + \sum_i s_{ai}(\eta(t'',h))h_i(t'')] ,$$

which gives the equation for $\chi_{ai}(t,t',h)$

$$\chi_{ai}(t,t',h) = \int_0^t dt''[\frac{\partial L_a(\eta)}{\partial \eta_{a'}} + \sum\nolimits_{i'} \frac{\partial s_{ai'}(\eta)}{\partial \eta_{a'}} h_{i'}(t'')]\Big|_{\eta=\eta(t'',h)} \chi_{a'i}(t'',t',h) + s_{ai}(\eta(t',h)) \qquad (21)$$

(it is assumed that $\eta_a(0,h)$ does not depend on $h_i(t)$). From here we have

$$\chi_{ai0}(t,h) = s_{ai}(\eta(t,h)) , \qquad (22)$$

and taking into account equation (1) the relation

$$\frac{\partial \chi_{ai}(t,t',h)}{\partial t'} = \int_0^t dt''[\frac{\partial L_a(\eta)}{\partial \eta_{a'}} + \sum\nolimits_{i'} \frac{\partial s_{ai'}(\eta)}{\partial \eta_{a'}} h_{i'}(t'')]\Big|_{\eta=\eta(t'',h)} \frac{\partial \chi_{a'i}(t'',t',h)}{\partial t'} +$$

$$+ \sum\nolimits_{a'} \frac{\partial s_{ai}(\eta)}{\partial \eta_{a'}}\Bigg|_{\eta=\eta(t',h)} [L_{a'}(\eta(t',h)) + \sum\nolimits_{i'} s_{a'i'}(\eta(t',h))h_{i'}(t')] , \qquad (23)$$

whence

$$\chi_{ai1}(t,h) = \sum\nolimits_{a'} \frac{\partial s_{ai}(\eta)}{\partial \eta_{a'}}\Bigg|_{\eta=\eta(t,h)} [L_{a'}(\eta(t,h)) + \sum\nolimits_{i'} s_{a'i'}(\eta(t,h))h_{i'}(t)] \equiv$$

$$\equiv b_{ai}(\eta(t,h)) + \sum\nolimits_{i'} c_{aii'}(\eta(t,h))h_{i'}(t) . \qquad (24)$$

Substitution (22) and (24) in (20) gives the relation

$$\Phi_i(t,u) = m_i \mathbf{F}(\mathrm{f}(t),u) +$$

$$+ \sum\nolimits_{a,i'} u_a \overline{\{n_{ii'0} s_{ai'}(\hat{\eta}(t)) - n_{ii'1}[b_{ai}(\hat{\eta}(t)) + \sum\nolimits_{i'} c_{aii'}(\hat{\eta}(t))\hat{h}_{i'}(t)] + O(\lambda^2)\} e^{\sum_{a'} \hat{\eta}_{a'}(t) u_{a'}}} ,$$

which according to (9) has the form of an equation for the function $\Phi_i(t,u)$

$$\Phi_i(t,u) = m_i \mathbf{F}(\mathrm{f}(t),u) +$$

$$+ \sum\nolimits_{a,i'} u_a \{[n_{ii'0} s_{ai'}(\frac{\partial}{\partial u}) - n_{ii'1} b_{ai'}(\frac{\partial}{\partial u})]\mathbf{F}(\mathrm{f}(t),u) + n_{ii'1} \sum\nolimits_{i''} c_{ai'i''}(\frac{\partial}{\partial u})\Phi_{i''}(t,u)\} + O(\lambda^2)$$

(the second identity from (7) is taken into account). The solution of this equation in the theory of perturbations in powers of λ is given by the formulas

$$\Phi_i(t,u)=\Phi_i^{(0)}(t,u)+\Phi_i^{(1)}(t,u)+O(\lambda^2),$$

$$\Phi_i^{(0)}(t,u)=m_i\mathbf{F}(\mathrm{f}(t),u)+\sum\nolimits_{i'}n_{ii'0}A_{i'}\mathbf{F}(\mathrm{f}(t),u)\ , \tag{25}$$

$$\Phi_i^{(1)}(t,u)=\sum\nolimits_{a,i'}n_{ii'1}u_a[-b_{ai'}(\frac{\partial}{\partial u})\mathbf{F}(\mathrm{f}(t),u)+\sum\nolimits_{i''}c_{ai'i''}(\frac{\partial}{\partial u})\Phi_{i''}^{(0)}(t,u)]$$

(operator A_i is defined in (9)). Thus, up to the second order in small parameter λ, i.e. in the approximation of the short time of correlation of the external random field, a closed equation for the generating function $\mathbf{F}(\mathrm{f}(t),u)$ is obtained, which describes the influence of fluctuations on the evolution of the system.

In the basic approximation, the time equation for the generating function $\mathbf{F}(\mathrm{f}(t),u)$ according to (7) and (25) has the form

$$\partial_t\mathbf{F}(\mathrm{f}(t),u)=e^{\sum_a \mathrm{f}_a u_a}e^{\mathbf{G}(g(t),u+\frac{\partial}{\partial\eta})}\sum\nolimits_a u_a L_a(\eta)\big|_{\eta=\mathrm{f}(t)}+\sum\nolimits_i m_i A_i(u)\mathbf{F}(\mathrm{f}(t),u)+ \\ +\sum\nolimits_{ii'}n_{ii'0}A_iA_{i'}\mathbf{F}(\mathrm{f}(t),u)]\ . \tag{26}$$

The equivalent equation separately for the cases of kinetic and hydrodynamic states was obtained in [3] by cumbersome calculations.

3. Generalized nonlinear fluctuation-dissipation theorem

Let discuss some consequences of equation (26) for the generating function $\mathbf{F}(\mathrm{f}(t),u)$. Concretize equation (26) first for the case of linear in the RDP inclusion of the external field in (1), i.e., for $s_{ai}(\eta)=\sum\nolimits_b s_{aib}\eta_b$ (s_{aib} are some coefficients). In terms of the average value of RDP $\mathrm{f}_a(t)$ and generating function of correlations $\mathbf{G}(g(t),u)$, it takes the form

$$\partial_t \mathrm{f}_a(t)=e^{\mathbf{G}(g(t),\frac{\partial}{\partial\eta})}L_a(\eta)\big|_{\mathrm{f}=\mathrm{f}(t)}+\sum\nolimits_b m_{ab}\mathrm{f}_b+\sum\nolimits_{abc}n_{ab,bc}\mathrm{f}_c$$

$$(\,m_{ab}\equiv\sum\nolimits_i s_{aib}m_i\,),$$

$$\partial_t\mathbf{G}(g(t),u)=\left[e^{\mathbf{G}(g(t),u+\frac{\partial}{\partial\eta})-\mathbf{G}(g(t),u)}-e^{\mathbf{G}(g(t),\frac{\partial}{\partial\eta})}\right]\sum\nolimits_a u_a\,L_a(\eta)\big|_{\eta\to\mathrm{f}(t)}+ \\ +\sum\nolimits_{ab}u_a m_{ab}\frac{\partial\mathbf{G}(g(t),u)}{\partial u_b}+\sum\nolimits_{abc}u_a n_{ab,bc}\frac{\partial\mathbf{G}(g(t),u)}{\partial u_c}+ \\ +\sum\nolimits_{abcd}u_a u_c n_{ab,cd}\left(\mathrm{f}_b(t)+\frac{\partial\mathbf{G}(g(t),u)}{\partial u_b}\right)\left(\mathrm{f}_d(t)+\frac{\partial\mathbf{G}(g(t),u)}{\partial u_d}\right) \tag{27}$$

$$(\,n_{ab,cd}\equiv\sum\nolimits_{ii'}n_{ii'0}s_{aib}s_{ci'd}\,).$$

Note, that a similar equation was obtained in [1] separately for kinetic and hydrodynamic states by cumbersome calculations in connection with the consideration of spatially inhomogeneous states. However, this is not necessary, because the kinetics of inhomogeneous systems is embedded in the developed theory after substitutions of the type

$$\eta_a \to \eta_\mu(x), \qquad g_{ab} \to g_{\mu\nu}(x,x'), \qquad u_a \to u_\mu(x), \qquad h_i \to h_\alpha(x);$$
$$\frac{\partial}{\partial \eta_a} \to \frac{\delta}{\delta \eta_\mu(x)}, \qquad \frac{\partial}{\partial u_a} \to \frac{\delta}{\delta u_\mu(x)};$$

$$\sum\nolimits_a \ldots = \sum\nolimits_\mu \int_V d^3x \ldots, \qquad \sum\nolimits_i \ldots = \sum\nolimits_\alpha \int_V d^3x \ldots.$$

Let now specify equation (26) for the case of inclusion of the external field in (1) through the constant, i.e., at $s_{ai}(\eta) \to s_{ai}$. Such an external field can be called additive one. In this case the average value of RDP $\mathrm{f}_a(t)$ and the generating function of correlations $\mathbf{G}(g(t),u)$ satisfy a simpler system of equations than (27)

$$\partial_t \mathrm{f}_a(t) = e^{\mathbf{G}(g(t),\frac{\partial}{\partial\eta})} L_a(\eta)\big|_{\mathrm{f}=\mathrm{f(t)}} + m_a \qquad (m_a \equiv \sum\nolimits_i s_{ai} m_i),$$

$$\partial_t \mathbf{G}(g(t),u) = \left[e^{\mathbf{G}(g(t),u+\frac{\partial}{\partial\eta}) - \mathbf{G}(g(t),u)} - e^{\mathbf{G}(g(t),\frac{\partial}{\partial\eta})} \right] \sum\nolimits_a u_a L_a(\eta)\big|_{\eta\to\mathrm{f}(t)} + \qquad (28)$$

$$+\sum\nolimits_{ab} n_{ab} u_a u_b \qquad (n_{ab} \equiv \sum\nolimits_{ii'} n_{ii'0} s_{ai} s_{bi'}).$$

According to (28), the time equations for binary and triple correlation functions are

$$\partial_t g_{ab} = e^{\mathbf{G}(g(t),\frac{\partial}{\partial\eta})} \left[\mathbf{G}_a(g(t),\frac{\partial}{\partial\eta}) L_b(\eta) + (a \leftrightarrow b) \right]_{\eta=\mathrm{f}(t)} + n_{ab}$$

$$\partial_t g_{abc} = e^{\mathbf{G}(g(t),\frac{\partial}{\partial\eta})} \left[\left(\mathbf{G}_b(g(t),\frac{\partial}{\partial\eta}) \mathbf{G}_c(g(t),\frac{\partial}{\partial\eta}) + \mathbf{G}_{bc}(g(t),\frac{\partial}{\partial\eta}) \right) L_a(\eta) + \right. \qquad (29)$$

$$\left. + (a \leftrightarrow b) + (a \leftrightarrow c) \right]_{\eta=\mathrm{f}(t)}$$

where it is denoted

$$\mathbf{G}_{a_1 \ldots a_n}(g(t),u) = \frac{\partial^n}{\partial u_{a_1} \ldots \partial u_{a_n}} \mathbf{G}(g(t),u). \qquad (30)$$

In the simplest approximation, the fluctuation kinetics of the system takes into account only binary field correlations and its state is described by the equations

$$\partial_t \mathrm{f}_a(t) = e^{\mathbf{G}_2(g(t),\frac{\partial}{\partial\eta})} L_a(\eta)\big|_{\mathrm{f}=\mathrm{f}(t)} + m_a ,$$

$$\partial_t g_{ab} = e^{\mathbf{G}_2(g(t),\frac{\partial}{\partial\eta})} \sum\nolimits_c \left[g_{ac}(t) \frac{\partial L_b(\eta)}{\partial\eta_c} + (a \leftrightarrow b) \right]_{\eta=\mathrm{f}(t)} + n_{ab} \qquad (31)$$

where the function is introduced

$$\mathbf{G}_2(g,u) = \frac{1}{2} \sum\nolimits_{ab} g_{ab} u_a u_b \, . \qquad (32)$$

In equilibrium, equations (31) give expressions for parameters m_a and n_{ab} by formulas

$$m_a = -e^{\mathbf{G}_2(g^{\mathrm{eq}},\frac{\partial}{\partial\eta})} L_a(\eta)\big|_{\eta=\mathrm{f}^{\mathrm{eq}}} ,$$

$$n_{ab} = -e^{\mathbf{G}_2(g^{\mathrm{eq}},\frac{\partial}{\partial\eta})} \sum\nolimits_c \left[g_{ac}^{\mathrm{eq}} \frac{\partial L_b(\eta)}{\partial\eta_c} + (a \leftrightarrow b) \right]_{\eta=\mathrm{f}^{\mathrm{eq}}} . \qquad (33)$$

The second of these formulas should be considered as our nonlinear generalization of the fluctuation-dissipation theorem ($\mathrm{f}_a^{\mathrm{eq}}$, g_{ab}^{eq} are equilibrium values of parameters f_a, g_{ab}). The same understanding of this theorem was proposed in our work [4].

Near the equilibrium in the quadratic approximation of the right part, equation (1) for RDP has the structure

$$L_a(\eta) = \sum\nolimits_b M_{a,b}(\eta_b - \mathrm{f}_b^{\mathrm{eq}}) + \frac{1}{2}\sum\nolimits_{bc} M_{a,bc}(\eta_b - \mathrm{f}_b^{\mathrm{eq}})(\eta_c - \mathrm{f}_c^{\mathrm{eq}}) \, . \qquad (34)$$

In this approximation, the time equations (31) for the parameters $\delta\mathrm{f}_a \equiv \mathrm{f}_a - \mathrm{f}_a^{\mathrm{eq}}$, $\delta g_{ab} \equiv g_{ab} - g_{ab}^{\mathrm{eq}}$ take the form

$$\partial_t \delta\mathrm{f}_a = \sum\nolimits_b M_{a,b}\delta\mathrm{f}_b + \frac{1}{2}\sum\nolimits_{bc} M_{a,bc}\delta g_{bc} ,$$

$$\partial_t \delta g_{ab} = \sum\nolimits_c (M_{b,c}\delta g_{ac} + M_{a,c}\delta g_{bc}) + \sum\nolimits_{cd} (g_{ac}^{\mathrm{eq}} M_{b,cd} + g_{bc}^{\mathrm{eq}} M_{a,cd})\delta\mathrm{f}_d \, . \qquad (35)$$

The fluctuation-dissipation theorem (33) and the expression for the mean field in approximation (34) are written as

$$h_{ab} = -\sum\nolimits_c (g_{ac}^{\mathrm{eq}} M_{b,c} + g_{bc}^{\mathrm{eq}} M_{a,c}) , \qquad m_a = -\frac{1}{2}\sum\nolimits_{bc} M_{a,bc} g_{bc}^{\mathrm{eq}} \, . \qquad (36)$$

Let apply the developed formalism for the case of hydrodynamics of a liquid, which is embedded in our general theory by substitutions

$$\eta_a \to \zeta_\mu(x) \to \zeta_{\mu k}, \qquad g_{ab} \to g_{\mu\nu}(x,x') \to g_{\mu k,\nu k'}, \qquad \sum\nolimits_a ... = \sum\nolimits_\mu \int_V d^3x ... \tag{37}$$

(with the Fourier transforms). Relevant substitutions are also made in formulas (35)

$$\begin{gathered} M_{a,b} \to M_{\mu,\nu}(x,x') \to M_{\mu k,\nu k'} \equiv V M_{\mu\nu}(k)\delta_{k',k}, \\ M_{a,bc} \to M_{\mu,\nu\lambda}(x,x',x'') \to M_{\mu k,\nu k'\lambda k''}, \quad M_{\mu k,\nu k'-q\lambda q} \equiv V M_{\mu,\nu\lambda}(k,q)\delta_{k',k}; \\ h_{ab} \to h_{\mu\nu}(x,x') \to h_{\mu k\nu k'} \equiv V h_{\mu\nu}(k)\delta_{k',-k}, \quad g^{\text{eq}}_{\mu k,\nu k'} \equiv V g_{\mu\nu}(k)\delta_{k',-k}. \end{gathered} \tag{38}$$

Equations of fluctuation hydrodynamics near equilibrium with quadratic approximation of nonlinearities in the right part of the equations of hydrodynamics $L_\mu(x,\zeta)$ are given by formulas

$$\begin{gathered} \partial_t \delta\zeta_{\mu k} = \sum\nolimits_\nu M_{\mu\nu}(k)\delta\zeta_{\nu k} + \frac{1}{2V}\sum\nolimits_{\nu\lambda q} M_{\mu,\nu\lambda}(k,q)\delta g_{\nu k-q\lambda q}, \\ \partial_t \delta g_{\mu k-q\nu q} = \sum\nolimits_\lambda \left[M_{\mu\lambda}(k-q)\delta g_{\lambda k-q\nu q} + M_{\nu\lambda}(q)\delta g_{\mu k-q\lambda q} \right] + \\ + \sum\nolimits_{\lambda\sigma} \left[g^{\text{eq}}_{\mu\lambda}(k-q) M_{\nu,\lambda\sigma}(q,k) + g^{\text{eq}}_{\nu\lambda}(q) M_{\mu,\lambda\sigma}(k-q,k) \right] \delta\zeta_{\sigma k}. \end{gathered} \tag{39}$$

These equations were the basis for the study of fluctuation effects in hydrodynamics in a known paper by Andreev [9]. In the same approximation, the fluctuation-dissipation theorem is given by the formula

$$h_{\mu\nu}(k) = -\sum\nolimits_\lambda \left[M_{\mu\lambda}(k) g^{\text{eq}}_{\lambda\nu}(k) + M_{\nu\lambda}(k) g^{\text{eq}}_{\nu\mu}(k) \right] \tag{40}$$

A simple hydrodynamic matrix $M_{\mu\lambda}(k)$ at small wave vector according to [8] has the form

$$M_{\mu\nu}(k) = -ik_n \frac{\partial \zeta_{\mu n}}{\partial \zeta_\nu} + k_n k_l \sum\nolimits_\lambda \Lambda_{\mu n,\lambda l} \frac{\partial Y_\lambda}{\partial \zeta_\nu} + ... \tag{41}$$

where $\Lambda_{\mu n,\nu l}$ is the matrix of kinetic coefficients, $\zeta_\mu : \varepsilon, \pi_n, \rho$ are the average values of energy densities, momentum and mass of the system, $\zeta_{\mu n}$ are the average values of the corresponding fluxes. These averages are calculated using the Gibbs distribution w

$$\begin{gathered} \zeta_\mu \equiv \mathrm{Sp} w \hat{\zeta}_\mu(0), \quad \zeta_{\mu n} \equiv \mathrm{Sp} w \hat{\zeta}_{\mu n}(0); \quad V g^{\text{eq}}_{\mu\nu}(k) = \mathrm{Sp} w \hat{\zeta}_{\mu k} \delta\hat{\zeta}_{\nu k} = -V \frac{\partial \zeta_\mu}{\partial Y_\nu} + ...; \\ w = \exp\left[\beta\Omega - \sum\nolimits_\mu Y_\mu \int_V d^3x \hat{\zeta}_\mu(x) \right], \end{gathered} \tag{42}$$

where Ω is thermodynamic potential, Y_μ: β, $-\beta\upsilon_n$, $-\beta(\mu - \upsilon^2/2)$ are the generalized thermodynamic forces (β, μ, υ_n are inverse temperature, chemical potential and liquid velocity). The cap in (42) indicates the microscopic values of the corresponding physical quantities. In these terms, the fluctuation-dissipation theorem is expressed by the formula

$$h_{\mu\nu}(k) = k_n k_l (\Lambda_{\mu n,\nu l} + \Lambda_{\nu n,\mu l}) + \dots, \qquad (43)$$

an analogue of which was discussed by Landau–Lifshitz [10].

4. Conclusions

Based on the general time equation of the method of the reduced description for some parameters $\eta_a(t)$, fluctuations caused by a random external field $\hat{h}_i(t)$, which is considered as a stationary Gaussian process, are investigated (random field functions are denoted by a cap). Cases of the field which is included through parameters of the reduced description and directly (additive field) are considered. The system is not specified and in a single approach the developed theory is suitable for the study of kinetic and hydrodynamic processes (in the simplest case, the former are described by a one-particle distribution function $f_p(x,t)$, and the latter by the densities of additive integrals of motion $\zeta_\mu(x,t)$). On this basis, the development of [3] is simplified, where such processes were investigated separately with an additional complication related to considering spatially inhomogeneous states. Fluctuations are described by us (besides the average values $f_a(t)$ of parameters $\hat{\eta}_a(t)$) by all their average products $f_{a_1 \dots a_n}(t)$ (to be short: by the fluctuations; $n \geq 2$) or the corresponding correlations $g_{a_1 \dots a_n}(t)$. For the generating functional of fluctuations $\mathbf{F}(f(t),u)$ (correlations $\mathbf{G}(g(t),\upsilon)$) in the approximation of the small time of the field correlation τ_0, a closed time equation (equation of the fluctuation kinetics) is derived. In this case, a small parameter $\lambda \equiv \tau_0 / T$ is introduced (T is the characteristic time of the system evolution) and the contributions of the main and first orders in λ are taken into account (in [3] only the main contribution was discussed). The derivation of the closed equation is based on the generalized Furutzu–Novikov theorem, the proof of which is simplified by us somewhat. An interesting problem about the possibility of derivation of a closed equation in higher approximations in λ is planning to be discussed in the next paper. It is noted that the consideration of spatially inhomogeneous states is reduced in the developed theory (as in the method of the reduced description in general) to the replacements of the type $\eta_a(t) \to \eta_\mu(x,t)$, $\sum_a \to \sum_\mu \int_V d^3x$. The approximations of the developed theory taking into account only binary correlations and states close to the equilibrium are considered. Such approximations have only been studied by previous researchers. It is proposed to introduce the generalized nonlinear fluctuation-dissipation theorem as the time equations of fluctuation kinetics in equilibrium state. These equations give an expression for the correlations of the random field through the equilibrium and nonequilibrium characteristics of the system. To illustrate, fluctuation hydrodynamics in the case of the additive field is considered and expressions for the field correlations through fluid kinetic coefficients like Landau–Lifshitz ones are obtained.

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