CORRELATIONS IN SUPERRADIANCE OF A TWO-DIMENSIONAL SYSTEM

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The process of cooperative spontaneous emission in a system of two-level emitters interacting via electromagnetic field is analyzed using the Bogolyubov reduced description method, which provides the possibility of studying the state of the generated field. Field macrostates are described with electric and magnetic field amplitudes and binary correlation functions of their components. Since the previously obtained set of differential equations describing the Dicke system evolution with considering field correlations is very cumbersome, we must find some way of simplifying the model to make it accessible for numerical modeling. The general picture of correlation development can be elucidated with using one- and two-dimensional models. The paper presents the set of evolution equations for Dicke system with fixed orientation of dipole moments of emitters forming a two-dimensional structure. The material equations and equilibrium correlations are discussed. The transition to the Cauchy problem for ordinary differential equations is proposed.

Keywords: Dicke model, reduced description, polarization current, material equations; binary correlations, Cauchy problem.

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1. Introduction

The problem of self-organization in a Dicke system of two-level electromagnetic emitters with generating the coherent emission pulse arouse persistent interest for more than six decades [1]. Basing of spontaneous emission, such Dicke superradiance process opens the way to coherent generation, which is alternative to the laser one and possible even in X-ray and γ range. At the same time its consideration made the significant contribution into the development of non-equilibrium quantum statistical physics and phase transition theory. The Bogolyubov reduced description method allows not only to calculate the pulse delay time proceeding from the emitter subsystem behavior but also to investigate field states. Deep research into the nature of quantum field states based on Glauber approach [2-4] stimulates our interest to the process of correlation growth in Dicke model. In 2004 Prof. Sokolovsky's group at Dnipro National University proposed a description of fluctuations in the emitter subsystem [5], and since 2008 the binary correlation functions of the electromagnetic field have been included into the list of reduced description parameters of the superradiant Dicke system [6]. We came to the phenomenon description using the ideas of electrodynamics of continuous media. The set of evolution equations for the model under consideration including correlation functions is presented in [7] in detail. However, the obtained complicated mathematical problem requires some simplification. The natural way of solving the problem is transition to the smaller number of dimensions. A one-dimensional system is the most attractive due to its simplicity and closeness to the real experimental situations applying oblong systems. Such approach has been discussed recently [8] but has not been implemented yet. At the same time material equations for this model case seem to be too artificial because of long-range correlations. So, it is interesting to study a two-dimensional system using the acquired experience. Great attention to two-dimensional systems in modern physics [9] is an additional reason for such a program. Hereafter we discuss the view of the set of evolution equations for correlation functions at fixed orientation of dipole moments, two-dimensional structure of emitter subsystem, field modes in the corresponding plane, and transition to the Cauchy problem for the numerical analysis of the correlation picture.

2. Evolution equations with binary correlations for a two-dimensional Dicke system

We proceed from of evolution equations describing the behavior of the system of two-level emitters interacting via electromagnetic field together with the generated field, which state in a certain moment is fixed by a set of amplitudes and simultaneous binary correlation functions [7]. Such functions include electric and magnetic field components and have a tensor character. Point emitter positions are unchanged. Emitter frequencies are equal ω, and at the last stage we exclude singularities with the assumption of ω sharp distribution near some value ω_0. If we pass to the one-dimensional case, i.e., consider the fixed orientation of the dipole moments of operation transition matrix elements along *Ox* axis and the wave vectors of the field modes along *Oz* axis [8], we have an evolution picture with 6 variables of field nature: $\langle E_1\rangle$, $\langle B_2\rangle$, (E_1E_1'), (E_1B_2'), (B_2B_2'), and linear density of emitter subsystem energy $\langle\varepsilon\rangle$ (this subsystem is regarded as locally equilibrium). The lower indices mean Cartesian components with usual correspondence $1\leftrightarrow x$, $2\leftrightarrow y$, $3\leftrightarrow z$. Averaged field components and ε depend on *z* for the one-dimensional system (or a cylindric system with small cross-section *S* [8]). Hatched variables correspond to the dependence on hatched coordinates. (ab') denotes a binary correlation function $(a(z)b(z')) = \frac{1}{2}\langle\{a(z),b(z')\}\rangle - \langle a(z)\rangle\langle b(z')\rangle$, $\{\ ,\ \}$ denotes operator anticommutator, and $\langle\ \rangle$ denotes the average of certain physical quantities with the statistical operator ρ in the reduced description form. The physical quantities are calculated in the second order of the perturbation theory in small parameter *d* – transition dipole moment. Just this approximation allows replacing the transversal electric field with the complete field [10]. Similar denotations will be used for a two-dimensional system of emitters.

Let us consider a great number of motionless emitters occupying some area in the *Oyz* plane with a known surface density *n* (*y*, *z*). The same picture will be observed in the case of a thin plane layer of thickness *h*. For simplicity, we assume the fixed orientation of the dipole moments of operation transition matrix elements along *Ox* axis. So, mode wave vectors are supposed to be localized in the *Oyz* plane. The list of reduced description parameters includes 10 ones: $\langle E_1\rangle, \langle B_2\rangle, \langle B_3\rangle, (E_1E_1'), (E_1B_2'), (E_1B_3'), (B_2B_2'), (B_2B_3'), (B_3B_3')$, and surface energy density of the emitter subsystem ε. The parameters depend upon the coordinates y and z (and y', z' for the binary correlation functions). Vector field rotors give derivatives with respect to the named coordinates as a contribution to the evolution equations. Calculations like [7] result in the set of equations

$$\begin{gathered}
\partial_t\langle E_1\rangle = c\frac{\partial\langle B_3\rangle}{\partial y} - c\frac{\partial\langle B_2\rangle}{\partial z} - \frac{4\pi}{h}\langle I_1\rangle,\ \partial_t\langle B_2\rangle = -c\frac{\partial\langle E_1\rangle}{\partial z},\ \partial_t\langle B_3\rangle = c\frac{\partial\langle E_1\rangle}{\partial y},\\
\partial_t(E_1E_1') = c\frac{\partial(B_3E_1')}{\partial z} - c\frac{\partial(B_2E_1')}{\partial z'} - \frac{4\pi}{h}(I_1E_1') + c\frac{\partial(E_1B_3')}{\partial y'} - c\frac{\partial(E_1B_2')}{\partial z'} - \frac{4\pi}{h}(E_1I_1'),\\
\partial_t(E_1B_2') = c\frac{\partial(B_3B_2')}{\partial y} - c\frac{\partial(B_2B_2')}{\partial z} - \frac{4\pi}{h}(I_1B_2') - c\frac{\partial(E_1E_1')}{\partial z'},\\
\partial_t(E_1B_3') = c\frac{\partial(B_3B_3')}{\partial y} - c\frac{\partial(B_2B_3')}{\partial z} - \frac{4\pi}{h}(I_1B_3') + c\frac{\partial(E_1E_1')}{\partial y'},\\
\partial_t(B_2B_2') = -c\frac{\partial(E_1B_2')}{\partial z} - c\frac{\partial(B_2E_1')}{\partial z'},\ \partial_t(B_2B_3') = -c\frac{\partial(E_1B_3')}{\partial z} + c\frac{\partial(B_2E_1')}{\partial y'},\\
\partial_t(B_3B_3') = c\frac{\partial(E_1B_3')}{\partial y} + c\frac{\partial(B_3E_1')}{\partial y'},\ \partial_t\langle\varepsilon\rangle = (I_1E_1) + \langle I_1\rangle\langle E_1\rangle - \frac{2d^2\omega_0^4}{c^3}\frac{n(y,z)}{h}.
\end{gathered} \tag{1}$$

In this set of equations I_1 means surface density of the one-component polarization current. Using such parameter for the emitter subsystem is natural because we neglect the dependence on *x*. For obtaining the closed set of evolution equations, it is necessary to express a medium reply through field parameters.

3. Material equations for the model under consideration

In the general case, we can use the only material equation for the medium of emitters, i.e., the dependence of volume polarization current density on E_n and $Z_n \equiv \mathrm{rot}_n B$, which has the form, characteristic for the presence of spatial dispersion:

$$I_n(\boldsymbol{x}) = \int_V d\boldsymbol{x}'[\sigma(\boldsymbol{x}-\boldsymbol{x}', \varepsilon(\boldsymbol{x}))E_n(\boldsymbol{x}') + c\xi(\boldsymbol{x}-\boldsymbol{x}', \varepsilon(\boldsymbol{x}))Z_n(\boldsymbol{x}')] \tag{2}$$

where the Fourier images of material coefficients are known [7]:

$$\sigma(\boldsymbol{k}, \varepsilon) = -\frac{2\pi d^2}{\hbar^2}\varepsilon w_\alpha(\omega_k), \quad \xi(\boldsymbol{k}, \varepsilon) = -\frac{4d^2}{\hbar^2}\varepsilon\, \mathrm{P}\int_0^{+\infty}\frac{w_\alpha(\omega)d\omega}{\omega^2-\omega_k^2}. \tag{3}$$

Hereafter bold symbols in function arguments denote vectors. According to our supposition, $w_\alpha(\omega) \to \delta(\omega-\omega_0)$, and expressions (3) transform into the following ones:

$$\sigma(\boldsymbol{k}, \varepsilon) = -\frac{2\pi d^2}{c\hbar^2}\varepsilon\delta(\frac{\omega_0}{c}-k), \quad \xi(\boldsymbol{k}, \varepsilon) = -\frac{4d^2}{c^2\hbar^2}\varepsilon\, \mathrm{P}\frac{1}{\left(\omega_0/c\right)^2 - k^2}. \tag{4}$$

We must find a spatial form of material coefficients for using (2) in the set (1). Thus, the Fourier transformations should be calculated

$$\sigma(\boldsymbol{x}-\boldsymbol{x}', \varepsilon(\boldsymbol{x})) = \frac{1}{V}\sum_{\boldsymbol{k}}\sigma(\boldsymbol{k}, \varepsilon(\boldsymbol{x}))e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}, \quad \xi(\boldsymbol{x}-\boldsymbol{x}', \varepsilon(\boldsymbol{x})) = \frac{1}{V}\sum_{\boldsymbol{k}}\xi(\boldsymbol{k}, \varepsilon(\boldsymbol{x}))e^{i\boldsymbol{k}(\boldsymbol{x}-\boldsymbol{x}')}. \tag{5}$$

In the thermodynamic limit the transition $\sum_{\boldsymbol{k}}... \overset{TL}{=} \frac{V}{(2\pi)^3}\int d^3\boldsymbol{k}...$ is valid. For our two-dimensional model this expression acquires the form $\frac{S}{(2\pi)^2}\int dk_2 dk_3...$ Since $k_1 = 0$, for arbitrary $\boldsymbol{x}-\boldsymbol{x}'$ we neglect the *x* component, fix the direction of this vector, and try to calculate the integral in the polar coordinates. Further on k and $|\boldsymbol{x}-\boldsymbol{x}'|$ mean modules of the corresponding vectors, and φ denotes angle between them. Integrating over φ gives

$$\int_0^{2\pi} e^{ik|x-x'|\cos\varphi}d\varphi = \frac{1}{2\pi}J_0(k|\boldsymbol{x}-\boldsymbol{x}'|) \tag{6}$$

where $J_0(k|\boldsymbol{x}-\boldsymbol{x}'|)$ denotes the Bessel function of the 1st kind [11]. Its behavior provides the physically convincing view of the material coefficients. Integrating over k is simple when we have $\delta(\frac{\omega_0}{c}-k)$ in the coefficient expression, and the conductivity takes the form

$$\sigma(x-x', \varepsilon(y,z)) = -\frac{d^2\varepsilon\omega_0 h}{c^2\hbar^2}J_0(\frac{\omega_0}{c}(x-x')) \tag{7}$$

Integrating over k for obtaining the second expression of (5) is reduced to the integral

$$\int_0^\infty \frac{kJ_0(k|\boldsymbol{x}-\boldsymbol{x}'|)}{k^2-a^2}dk\,, \tag{8}$$

which exists in Cauchy sense only if $\operatorname{Im} a \neq 0$, then it is expressed through the Bessel function of an imaginary argument of the 2nd kind [11]. Thus, to obtain a meaningful result, damping should be taken into account. An additional comment is necessary in connection with using the surface densities. To reflect this fact, we introduce the factor h into the material coefficient expressions. In our case we come to such view of the material equation for surface current density

$$I_1(y,z)=\int_S dy'dz'\left[\sigma(\boldsymbol{x}-\boldsymbol{x}',\varepsilon(y,z))E_1'+c\xi(\boldsymbol{x}-\boldsymbol{x}',\varepsilon(y,z))\left(\frac{\partial B_3'}{\partial y'}-\frac{\partial B_2'}{\partial z'}\right)\right] \tag{9}$$

Notice that in (9) $|\boldsymbol{x}-\boldsymbol{x}'|=\sqrt{(y-y')^2+(z-z')^2}$, σ is given by (7), ξ is obtained proceeding from (8).

The next problem is obtaining material equations for the correlation functions. Here we make use of the relationships substantiated in [7]:

$$(E_nI_l')^{(2)}=(E_nI_l'^{(2)})+S_{nl}(\boldsymbol{x}-\boldsymbol{x}',n(\boldsymbol{x}'))\,,\quad (B_nI_l')^{(2)}=(B_nI_l'^{(2)})+T_{nl}(\boldsymbol{x}-\boldsymbol{x}',n(\boldsymbol{x}'))\,, \tag{10}$$

in which the last terms originated due to the non-commutativity of the operators E_n and B_n. We once more must remind that our consideration is restricted with the reduced description parameters in the second order of perturbation theory (pointed out with the higher indices $^{(2)}$ in (10)) and expression (9) is valid just in this order. Now it is possible to apply (9) in the set (1) and right-hand sides of (10). The calculations like performed for the 3-dimensional system give for the last terms of (10) expressions that differ from 0 in several cases, i.e., S_{11}, T_{21}, T_{31}. Pay attention that correlation functions are symmetrical in respect to transposing correlated variables and arguments. So, we can restrict ourselves with specified examples

$$S_{11}(\boldsymbol{x}-\boldsymbol{x}',n(\boldsymbol{x}'))=-\frac{d^2\omega_0^3}{4\pi^2c^2}n(y',z')\frac{1}{h^2}J_0\left(\frac{\omega_0}{c}|\boldsymbol{x}-\boldsymbol{x}'|\right), \tag{11}$$

$$T_{21}(\boldsymbol{x}-\boldsymbol{x}',n(\boldsymbol{x}'))=-\frac{4\pi d^2\omega_0^2}{ch^2}n(y',z')\frac{\partial}{\partial z}\mathrm{P}\int_0^\infty J_0\left(k|\boldsymbol{x}-\boldsymbol{x}'|\right)\frac{kdk}{\left(\omega_0/c\right)^2-k^2}, \tag{12}$$

$$T_{31}(\boldsymbol{x}-\boldsymbol{x}',n(\boldsymbol{x}'))=\frac{4\pi d^2\omega_0^2}{ch^2}n(y',z')\frac{\partial}{\partial y}\mathrm{P}\int_0^\infty J_0\left(k|\boldsymbol{x}-\boldsymbol{x}'|\right)\frac{kdk}{\left(\omega_0/c\right)^2-k^2}. \tag{13}$$

The same technique is applied in the last equation of the set (1) for $\partial_t\langle\varepsilon\rangle$ where (I_1E_1) means just $(I_1^{(2)}E_1)$. Thus, we obtain the possibility of rewriting the set of evolution equations in terms of one-moment correlation functions.

4. The final set of equations and prospects of its numerical solving

Substituting the obtained expressions for current density and binary correlation functions including currents calculated in the 2nd order of perturbation theory current into the differential equations of evolution should be done with considering the field nature of the selected variables. Integral representations reflecting the presence of spatial dispersion in the system of emitters result in integrodifferential kind of final equations. Since the set is rather complicated, it seems to be justified giving one of the equations as an example. Let us take the 5th equation of the set (1). It is written down for an arbitrary pair of points $\boldsymbol{x}=(y,z)$ and $\boldsymbol{x}'=(y',z')$. Integrating variables in the formula for I_1 may be denoted with double scratched letters (y'',z''). We come to the equation.

$$\partial_t(E_1B_2')=c\frac{\partial(B_3B_2')}{\partial y}-c\frac{\partial(B_2B_2')}{\partial z}-c\frac{\partial(E_1E_1')}{\partial z'}-$$
$$-\frac{4\pi}{h}\iint dy''dz''\left[\sigma(\boldsymbol{x}-\boldsymbol{x}'',\varepsilon(y,z))\left(E_1''B_2'\right)-c\xi(\boldsymbol{x}-\boldsymbol{x}'',\varepsilon(y,z))\left(\frac{\partial(B_3''B_2')}{\partial y''}-\frac{\partial(B_2''B_2')}{\partial z''}\right)\right]- \quad (14)$$
$$-4\pi T_{12}(\boldsymbol{x}'-\boldsymbol{x},n(y,z))$$

The last term describes equilibrium correlations and does not depend on the emitter subsystem state. To calculate it, we apply the relation $(I_1B_2')^{(2)}=(B_2'I_1^{(2)})+T_{21}(\boldsymbol{x}'-\boldsymbol{x},n(y,z))$ and take into account using surface densities in our consideration. Analogous structures are present in the other evolution equations. Their number is enough because scratched and unscratched variables have the same domain. The way to the numerical analysis of our problem proves to be attractive due to expressing the coefficients in tabulated functions, namely Bessel ones (in the one-dimensional case we faced with trigonometric functions). The next step to simplification is studying the initial stage of excited emitter system evolution when its energy density remains almost without changes. Thus, we come to the set of 9 integrodifferential equations instead of 5 in the one-dimensional case. Nevertheless, it is possible to simulate the process using the program package *Mathematica*. Our subject is integrodifferential equations with partial derivatives. Solving it demands some border conditions, that is not physically substantiated. Moreover, we use the formulas for the unlimited medium and neglect boundary effects. At the same time, the scheme of solving a boundary value problem requires constructing an expansion in basis at each step and hence too much time. Our attempts to apply *Mathematica* were not successful.

Now the idea of using for modeling the Cauchy scheme when only dependence of functions on time is considered. Solving Cauchy problem for ordinary differential equations is the mainstream in the process analysis [12]. Early research into fluctuation role in Dicke process was based on such approach [13]. Now we must go to the totality of functions given on the lattice. Their spatial derivatives can be represented according to a difference scheme. For example, when the lattice function is set, the representation of the derivative $u_i'\approx\frac{1}{12h}(-u_{i+2}+8u_{i+1}-8u_{i-1}+u_{i-2})$ is applicable up to the terms $O(h^4)$ [12] where h is the lattice step. In this way, we expect to obtain the numerical model of correlation development in the Dicke process. Obviously, the problem of function smoothness and the required number of points should be analyzed.

4. Conclusions

The paper presents the set of integrodifferential equations describing the evolution of two-dimensional Dicke system with taking into account binary correlations of electromagnetic field. The problem is considered with great simplifications: only modes in the plane of the emitter subsystem are considered, and the polarization of emitters is assumed to be fixed. This makes the number of parameters acceptable. The set becomes closed due to the found form of material coefficients including cylindric functions, but the divergence of some coefficients requires a damping process. Comparing with one-dimensional systems, the behavior of conductivity coefficient is more physically convincing. Since direct use of math packages fails for a boundary problem, transition to the Cauchy problem with difference representation for derivatives can be proposed.

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