LOCALIZATION OF PLANE WAVE EXPANSION USING THE METHOD OF QUASISOLUTION SEARCHING

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The localization of the plane wave expansion is considered for the analysis of wave beam transformations on transversely inhomogeneous structures. To implement compact localization in the spatial domain, it is proposed to use a system of non-intersecting rectangular windows. Problems of discrete realization of local plane wave expansion are analyzed. An algorithm for the numerical implementation of the localization of the plane wave expansion based on the quasisolution search method is proposed. Its advantage over the algorithm based on the discrete Fourier transform is demonstrated. The efficiency of its use has been demonstrated by the example of analyzing the wave beam reflection from the secondary aperture with transverse inhomogeneity.

Keywords: plane wave expansion, local plane wave spectrum, wave beam, antenna radiation, windowed Fourier transform, method of quasisolution searching, parametric spectral analysis.

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1. Introduction

Plane wave expansion (PWE) is widely used for considering wave propagation processes [1, 2]. The set of plane waves (PW) forms a convenient orthogonal system of functions with continuous spectrum in the wave number domain for the expansion of arbitrary field [1, 2]. In this case, the operational relations for determining the amplitudes of plane waves can be reduced to algebraic ones [2]. The relative simplicity of analyzing the behavior of each of the harmonic waves provides range of opportunities for analyzing complicated wave process.

However, for some problems the disadvantage of the plane wave expansion is its globality, since it is determined by the field spatial distribution along the full real axis [3, 4]. On the other hand, the efficiency of this approach can be increased by going over to obtaining a spectral representation of the radiation field for a limited fragment of the aperture [5]. The localization of the expansion should allow extracting a set of dominant directions of wave propagation for each specific limited fragment of the field distribution under consideration.

The problem of globality manifests itself especially clearly in the numerical implementation of the plane wave expansion for processing experimental data [4]. In this case, the plane wave spectrum (PWS) can be numerically obtained using a discrete Fourier transform (DFT), which involves the use of discrete samples of the field distribution, specified on a finite support [5]. The size of the used interval is determined by the degree of divergence of the wave beam under consideration and the distance to the plane in which the transverse distribution of the field is supposed to be restored [4]. At small sizes of the secondary aperture, the size of the interval for the DFT of spatial distribution must be chosen many times larger than the size of this aperture, especially in the case of an asymmetric phase distribution on it. The number of discrete samples increases in proportion to the size of the interval and can be significant, that complicates calculations.

For implementation of the possibility of localization of features of the radiation field in several works, for example in [3, 6], it was proposed to use a window expansion on plane waves with a Gaussian function as the window. The main disadvantage of this approach is the non-orthogonality of the basis functions. Also, for the window expansion on plane waves, there is a problem of effective simultaneous localization in both spatial and spectrally conjugate field. Ways to overcome these problems and localization of the plane wave expansion using the method of quasisolution searching are discussed in this paper.

2. Localization of plane wave expansion using a window

The localization of the features of the transformed function can be realized by introducing a moving window function having a compact support into the Fourier transform. Use of the window function makes the result of the transformation dependent on the window coordinate. On this basis, in a number of works, for example, in [6], to localize the features of the radiation field $u(x,z)$, it was proposed to use a window expansion on plane waves

$$U(x,\kappa)=\int_{-\infty}^{\infty} w(\chi-x)u(\chi)e^{j\kappa\chi}d\chi=\frac{1}{2\pi}\int_{-\infty}^{\infty}\hat{w}(\xi-\kappa)U(\xi)e^{-j(\xi-\kappa)x}d\xi\,, \tag{1}$$

where $U(x,\kappa)$ is local spectrum (LS) of plane waves at point x, $\kappa=\sqrt{k^2-k_z^2}$ is the transverse component of the wave vector k, $w(x)$ is spatial window, and $\hat{w}(\kappa)$ is the corresponding window in the spectral domain, which is defined as

$$\hat{w}(\kappa)=\int_{-\infty}^{\infty} w(x)\exp(j\kappa x)dx\,. \tag{2}$$

When using the delta function as the window function $w(x)=\delta(x)$ for the utmost localization in the spatial domain, the local spectrum of the PW takes the form $U(x,\kappa)=u(x)\exp(j\kappa x)$. Along the spectral coordinate κ, it is homogeneous, and its amplitude does not depend on κ, and along the spatial coordinate x, it is determined by the distribution $u(x)$. The kind of the local PWS at utmost localization corresponds to the Huygens principle: each point of the wavefront is a local point source (a source of a local cylindrical wave).

In another limiting case for the window function $w(x)=\exp(-j\kappa x)$ which is homogeneous in the full domain of x and which provides the utmost localization in the spectral-conjugate region $\hat{w}(\kappa)=\delta(\kappa)$, the localization along x is lost for the local spectrum of PW

$$U(x,\kappa)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\delta(\xi-\kappa)U(\xi)\exp[-j(\xi-\kappa)x]d\xi=U(\kappa)\,. \tag{3}$$

In this case for any x the local PWS is determined by the form of the global spectrum of PW.

The Gabor transform with the Gaussian function as the window is widely used in the analysis of antenna radiation [7]. At the expense of choosing the Gaussian function as the window one, simultaneous localization of the basis function is achieved in both the spatial and spectral-conjugate domains, and the localization rectangle (Heisenberg rectangle) has the smallest area in comparison with other windows. The use of Gabor transform is also justified by the fact that the basis function has a physical interpretation in the form of Gaussian beam. The main disadvantage of the Gabor transform is the non-orthogonality of the basis functions.

Compact localization in the spatial domain is provided by a rectangular window, for which $w(\chi-x)=0$ for $\chi\notin[x-w/2,\,x+w/2]$. When using it, the local PWS will be defined as

$$U(x,\kappa)=U_x(\kappa)=\int_{x-w/2}^{x+w/2} u(\chi)\exp(j\kappa\chi)d\chi \tag{4}$$

where w is the width of the window. The system of non-intersecting rectangular windows with centers on the grid $x_i = iw$, $i \in (-\infty,\infty)$ is orthogonal, the window is specified on the finite support and the transition from local to global representation is implemented as simply as possible by summation [5]. In this case, the formed system of local spectra of plane waves $\{U_i\}_{i\in(-\infty,\infty)}$ determines the initial global spectrum of plane waves $U(\kappa)$ by its sum

$$U(\kappa)=\sum_{i=-\infty}^{\infty}U_i(\kappa)=\sum_{i=-\infty}^{\infty}\int_{(i-1/2)w}^{(i+1/2)w}u(\chi)\exp(j\kappa\chi)d\chi=\int_{-\infty}^{\infty}u(\chi)\exp(j\kappa\chi)d\chi \quad (5)$$

where the local spectrum $U_i(\kappa)$ is defined as

$$U_i(\kappa)=\int_{(i-1/2)w}^{(i+1/2)w}u(\chi)\exp(j\kappa\chi)d\chi\,. \quad (6)$$

If, when considering transformations at the boundary of a transverse-inhomogeneous structure, it is assumed that the rate of change of properties along the boundary is insignificant and within the window $[(i-1/2)w, (i+1/2)w]$ the local spectrum of secondary PW can be defined as $V_i(\kappa) = R_i(\kappa)U_i(\kappa)$, then the transverse distribution of the secondary wave can be found using

$$v(x)=\sum_{i=-\infty}^{\infty}v_i(x)=\sum_{i=-\infty}^{\infty}\int_{-\infty}^{\infty}R_i(\kappa)U_i(\kappa)\exp(-j\kappa x)d\kappa\,. \quad (7)$$

This makes it possible to determine the radiation field of the secondary aperture formed by a section of finite length D of the boundary under consideration, using the summation

$$v(x)=\sum_{i=0}^{I-1}\int_{-\infty}^{\infty}R_i(\kappa)U_i(\kappa)\exp(-j\kappa x)d\kappa\,, \quad (8)$$

where $I = D/w$ is the number of windows covering the secondary aperture, w is the window length.

3. Discrete implementation of local expansion in terms of plane wave

Under the numerical analysis of the radiation field, it is assumed that the discrete distribution of the sources $\mathbf{u} = \{u_0, u_1,\ldots, u_{N-1}\}$ on the grid $x_n = x_0 + n\Delta x$ is known. Here $\Delta x = D/N$ is the step of spatial sampling, D is the length of the analyzed segment of the field distribution, $n = 0,1,\ldots,N-1$. In this case, the segment of localization of the spectrum of plane waves with the number of samples $L \le N$ and length $w = L\Delta x$ will be determined by the number i of its initial sample x_i as $\mathbf{u}_i = \{u_i, u_{i+1},\ldots, u_{i+L-1}\}$.

If the analyzed sequence of spatial field samples is considered as a sequence $\mathbf{u}=\left[\mathbf{u}_0,\mathbf{u}_L,\ldots,\mathbf{u}_{(I-1)L}\right]$ of its non-intersecting parts $\mathbf{u}_{iL}=\left\{u_{iL},u_{iL+1},\ldots,u_{(i+1)L-1}\right\}$ which have equal length, then in this case the local PWS can be determined using the discrete Fourier transform of the field distribution on the segment of the localization

$$U_m^{iL}=\frac{1}{N}\sum_{l=0}^{L-1}u_{iL+l}\exp(j\kappa_m x_{iL+l})\,, \quad (9)$$

where $L = N/I = w/\Delta x$ is the number of samples in the window, $\Delta x = D/N$ is the spatial sampling step, U_m^{iL} is m-th component of the local PWS of the i-th part of the sequence. However, the discrete grid in the spectral-conjugated region $\kappa_m = (m - N/2)2\pi/D$ does not correspond to the length-limited spatial grid of the window $x_{iL+l} = x_{iL} + lD/N$, whose length is $w = D/I$ instead of the length D required for coincidence. Therefore, in order to use the DFT to calculate the local spectrum of plane waves, it is necessary either to switch to a more sparse grid $\kappa_\mu = (\mu I - N/2)2\pi/D$ in the spectral-conjugate region, or padding the transformed subsequence $\mathbf{u}_i$ with the corresponding number of zeros at the beginning and at the end to apply the DFT to the full-length sequence.

The problem in determining the local PWE using the DFT is that as the number of windows I increases to increase the detail, the window length $w = D/I$ decreases, and the localization in the spectral-conjugate domain worsens by I times, which manifests itself in the corresponding spreading of the local spectrum (fig. 1-2). If the DFT is applied directly to the window samples without padding zeros, then with increasing number of windows I the window sample number L decreases. This leads to a decrease in the number of samples in the spectrum while its width remains unchanged and, accordingly, to a decrease in the number of undamped plane waves $L_p = 2L\Delta x/\lambda + 1$, which are part of the local spectrum of plane waves. With the reverse reconstruction of the secondary wave, this can lead to a decrease in the detail of the secondary local spectra. If the sampling step is very small and the condition $\Delta x = \lambda/2L$ is satisfied for the selected partitioning of the analyzed interval into windows, then the local spectrum of plane waves will contain only one undamped component corresponding to the zero propagation angle. The condition $D = I\lambda/2$ is similar when the window length $w = L\Delta x$ is equal to half the wavelength λ.

If propagation direction $\arcsin\left(\kappa^i_{\max}/k\right)$ prevails in the area under consideration, then it is desirable that $\Delta\kappa_L \le \kappa^i_{\max}$ or $L \ge \lambda^i_{\max}/\Delta x$ are satisfied, where $\lambda^i_{\max}$ is the spatial period of the most significant change in this area. To overcome this problem, when obtaining the local PWS it is necessary to apply the DFT to the window samples with zeros appended. In this case, the number of undamped waves in the local spectrum increases I times up to $N_p = 2N\Delta x/\lambda + 1$, which is typical for the PWS obtained over the full sequence $\mathbf{u}$.

Let us demonstrate the considered features as well as the dependence of the local PWS obtained with the DFT on the number of windows I using an example. Fig. 1 and 2 demonstrate the LS for the field distribution created by the primary aperture $a/\lambda = 1$ on the distance $z/\lambda = 5$ at the secondary aperture $D/\lambda = 20$ when $N = 1000$. In Fig. 1 for the case $I = 8$ ($w/\lambda = 2.5$), the LS distinguish, in addition to the main one, several other directions of propagation, and in Fig. 2 for the case $I = 20$ ($w/\lambda = 1$) the LS localize only one direction of propagation. The discretization error of the PWE implemented by the DFT is determined by the number of undamped plane waves in the PWS $N_p = 2N\Delta x/\lambda + 1$.

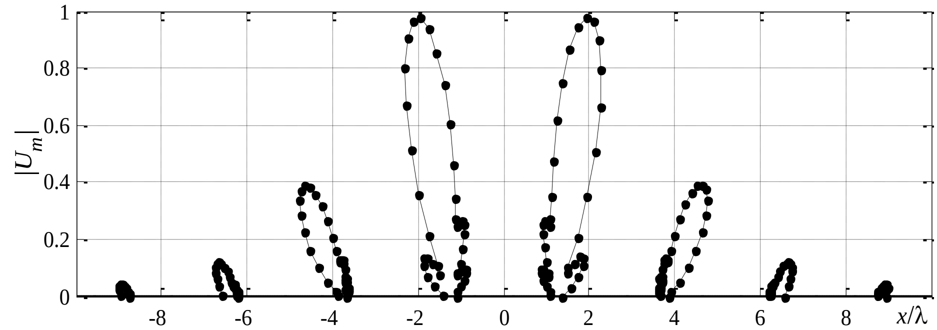

Fig. 1. Local spectra of plane waves for $I = 8$ ($w/\lambda = 2.5$).

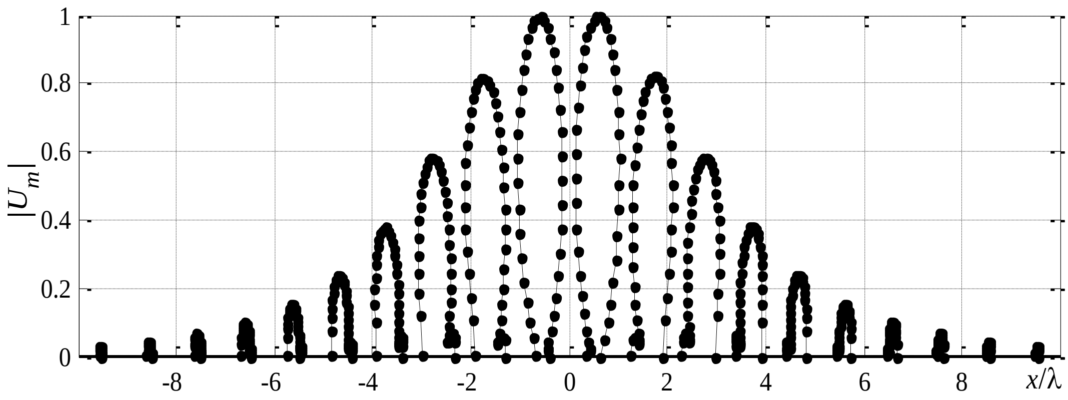

Fig. 2. Local spectra of plane waves for *I* = 20 (*w*/λ = 1).

4. Using the concept of quasisolution to localize the plane wave expansion

The local PWS for the rectangular window is determined by convolution with the spectral image of a rectangular window $\hat{w}(\kappa)=\mathrm{sinc}(\kappa w/2)$, which significantly worsens localization in the spectral domain. To overcome this problem, the local spectrum of plane waves $U(x,\kappa)$ will be defined as one of the solutions that provide the minimum of the functional

$$\rho[U(x,\kappa)]=\int_{x-w/2}^{x+w/2}\left|u(x)-\int_{-\infty}^{\infty}U(x,\kappa)\exp(-j\kappa x)d\kappa\right|^2 dx\,. \quad (10)$$

For some physical problems it is quite possible to confine oneself to the search for a quasisolution on a compact set by minimizing

$$\rho(\mathbf{U}_x,\boldsymbol{\kappa}_x)=\int_{x-w/2}^{x+w/2}\left|u(x)-\sum_{m=1}^{M}U_m^x\exp\left(-j\kappa_m^x x\right)\right|^2 dx\,, \quad (11)$$

where the local spectrum for the segment $[x - w/2, x + w/2]$ is determined by the values $\mathbf{U}_x=\{U_1^x,U_2^x,\ldots,U_M^x\}$ on discrete samples $\boldsymbol{\kappa}_x=\{\kappa_1^x,\kappa_2^x,\ldots,\kappa_M^x\}$. For the case of a discrete distribution of sources, the local spectrum of plane waves will be determined by the values $\mathbf{U}_i=\{U_1^i,U_2^i,\ldots,U_M^i\}$ at discrete samples $\boldsymbol{\kappa}_i=\{\kappa_1^i,\kappa_2^i,\ldots,\kappa_M^i\}$ that provide the minimum of the function [8]:

$$\rho(\mathbf{U}_i,\boldsymbol{\kappa}_i)=\sum_{l=0}^{L-1}\left|u_{i+l}-\sum_{m=1}^{M}U_m^i\exp\left(-j\kappa_m^i x_{i+l}\right)\right|^2 = \left\|\mathbf{u}_i-\mathbf{u}_i^M\right\|^2. \quad (12)$$

In this case, the quasisolution searching method (QSM) is reduced to determining the parameters of exponential model $\mathbf{u}_i^M=\mathbf{u}_i^M(\mathbf{U}_i,\boldsymbol{\kappa}_i)=\{u_i^M,u_{i+1}^M,\ldots,u_{i+L-1}^M\}$ with the components

$$u_{i+l}^M=\sum_{m=1}^{M}U_m^i\exp\left(-j\kappa_m^i x_{i+l}\right) \quad (13)$$

approximating the field distribution for the i-th localization segment (here $l = 0,1,\ldots,L-1$) and is a more efficient way to localize the spectrum than using the DFT.

The advantage of spectrum localization by the quasisolution searching method over the DFT-based method is that it gives not the spectrum on a uniform grid, but the amplitudes $\mathbf{U}_i = \{U_1^i, U_2^i, \ldots, U_M^i\}$ of the main components of the plane wave spectrum and its locations $\boldsymbol{\kappa}_i = \{\kappa_1^i, \kappa_2^i, \ldots, \kappa_M^i\}$. The shorter the support length w on which the field distribution $\mathbf{u}_i$ is approximated, the lower the model order M can be used. In the limit at $M = 1$, we pass to the geometric optics approximation. For the numerical realization of the QSM, methods of parametric spectral analysis can be used, which make it easy to implement the sliding version, when the area of consideration $\mathbf{x}_i = \{x_i, x_{i+1}, \ldots, x_{i+K-1}\}$ moves sequentially along the analyzed distribution $\mathbf{u}$ to determine the dynamics of changes of the local PWS in the transverse direction of the radiation field.

This advantage is clearly demonstrated by the results presented in fig. 3. It presents the values of QSM local estimates $\mathbf{U}_i$ and $\boldsymbol{\kappa}_i$ in the form of a discrete angular spectrum for the field distribution from the previous example and $I = 100$ ($L = 10$, $w/\lambda = 0.2$). In this case, the model order $M = 3$ was used. The components of the spectrum are shown as an arrow diagram for each location of the window (since the distribution is symmetrical for better visualization only half of it is shown), the tilt of the components U_m^i is determined by the ratio κ_m^i / k, the components for different m differ in different shades of gray.

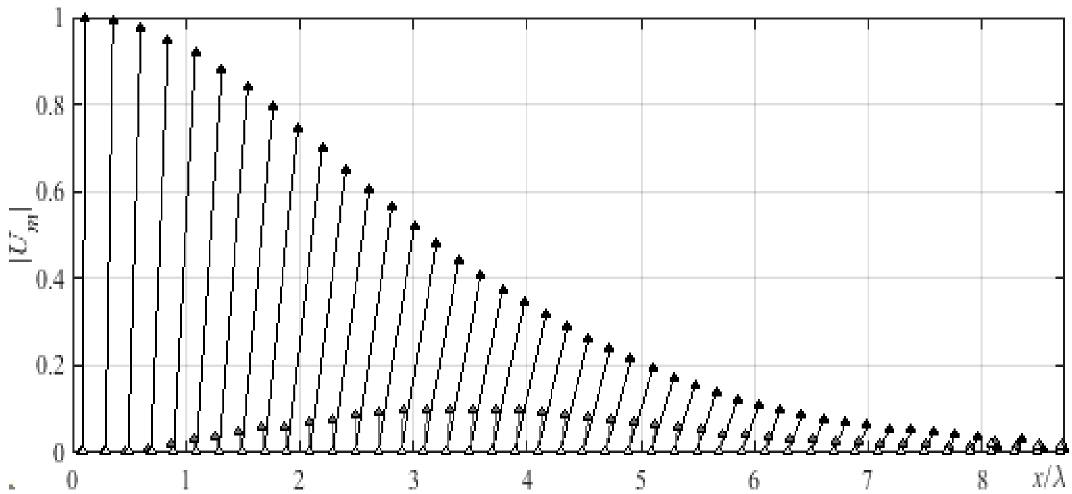

Fig. 3. Estimates of the quasisolution searching method for I = 100 (w/λ = 0.2)

The advantages of using the quasisolution searching method for the localization of the PWS are manifested when considering wave transformations that change the shape of the beam, for example, for reflection from the structure with the inhomogeneous angular spectrum of the reflection coefficient $R(\kappa)$. The example of reconstructing the spatial distribution of the wave reflected from a segment of the interface between media with permittivity $\varepsilon_1 = 1$ and $\varepsilon_2 = 2$ ($D/\lambda = 10$) using the fast Fourier transform (FFT) and the QSM is shown in fig. 4 (for comparison, the distribution of the reflected field from unlimited structure is shown with triangular markers). The distribution of the incident field was taken from the previous example with using the partition with the number of windows $I = 50$. The use of the QSM led to the coincidence of the reconstructed characteristic $\mathbf{v}_M$ with the true $\mathbf{v}$ with graphic accuracy. The reconstruction error when using the FFT is due to spectrum "spreading" for harmonic components with spatial frequencies that do not coincide with the grid κ_m.

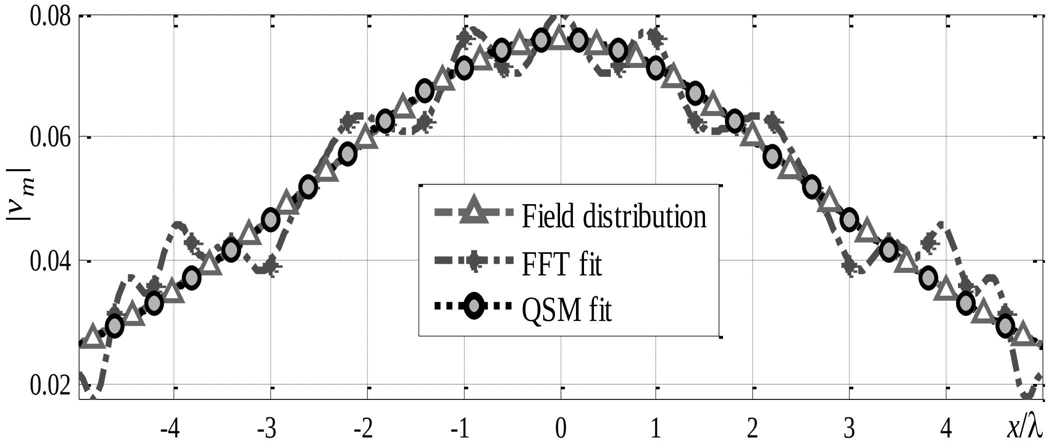

Fig. 4. Reconstruction of the reflected field from the local spectra of plane waves.

The efficiency of the proposed approach was tested on the example of the transversely inhomogeneous structure as a layer with Gaussian change in the permittivity along the transverse coordinate $\varepsilon(x)=1+(\varepsilon_2-1)e^{-x^2/(D/4)^2}$, electrical thickness $\sqrt{\varepsilon_2}d/\lambda=0.5$ and transverse size D/λ = 10 for ε_2 = 2. The result of reconstructing the distribution of the reflected field in the plane of the primary aperture is shown in fig. 5. For comparison, the line with triangular markers (with title "Global PWE") demonstrates the result of using the global PWE (by the values of ε averaged along *x*), which cannot consider the transverse inhomogeneity.

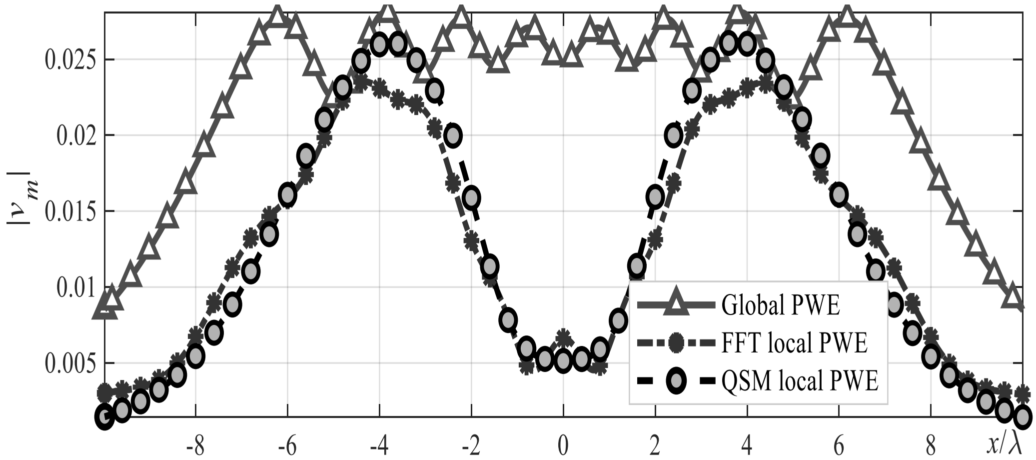

Fig. 5. Distribution of the reflected field for the transversely inhomogeneous structure.

5. Conclusion

Using the DFT to compute the plane wave spectrum limits the locality of the analysis. More efficient for the localization of the PWS is the application of the quasisolution searching method. Its use when decreasing the window size provides an advantage over the DFT-based algorithm. The proposed approach has confirmed its effectiveness by the example of analyzing the reflection from the secondary aperture with the transverse inhomogeneity.

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