PROPAGATION OF THE WEYL SPINOR PACKET THROUGH A POTENTIAL BARRIER

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We investigate the propagation of the Weyl spinor wave packet through a rectangular potential barrier. The coefficients of the reflection and transition are calculated. It is discovered that the Hartman effect, known in nonrelativistic case, is absent. The spinor could not exceed speed of light in the quantum tunneling. This is due to the simple dispersion relation for the relativistic massless particle.

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1. Introduction

Transition of particles through different potential barriers is a classical problem of quantum mechanics having either theoretical or practical interests. It demonstrates basic properties of quantum motion which are wide investigated in the literature, see, for instance, for references [1]. A practical interest has different sort wave pulses or packets propagating in different technical devices. In contrast to well studied case of plane waves, for the packets interesting new phenomena have been observed. One them is so-called the Hartman effect [2] consisting in a possibility to exceed the speed of light in the transition. This is not in contradiction to the main physical principles, but is the consequence of quantum superpositions for plane waves.

The Hartman effect has been discovered in nonrelativistic quantum mechanics for massive particles. So, it is interesting to consider the case of relativistic particles with different dispersion laws. The most simple and important case is one-half massless spinor with simple dispersion relation $k_0 = c|\vec{k}|$, where c is speed of light, $\vec{k}$ is particle momentum, k_0 is energy. This is problem investigated below. In next section we consider general properties of Weyl's spinors relevant to our problem. In section 3 we carry out all our calculations. In section 4 we summarize the results obtained. In Appendix we present mathematical information used in the main text.

2. Hamiltonian of the system

Motion of massless spin one-half particle is described by the Weyl equation

$$i\sigma^{\mu}\partial_{\mu}\chi = 0 \tag{1}$$

where σ^0- unit matrix, $\sigma^1, \sigma^2, \sigma^3-$ the Pauli matrixes, $\chi-$ is two-component spinor. For one-dimensional case, assuming motion along the x axis, we have:

$$i\sigma^0\frac{\partial\chi}{\partial t} + i\sigma_1\frac{\partial\chi}{\partial x} = 0. \tag{2}$$

We will search for the free particle solution in the form $\chi = \chi_0 e^{-iEt}$, and get

$$E = -\sigma^1\partial_x\chi_0, \tag{3}$$

or in the components

$$\begin{aligned} E\chi_1 &= -i\partial_x\chi_2, \\ E\chi_2 &= -i\partial_x\chi_1. \end{aligned} \tag{4}$$

Expressing χ_2 from the second equation and substituting it in the first we obtain

$$\begin{aligned} E^2\chi_1 &= -\frac{\partial^2}{\partial x^2}\chi_1, \\ \chi_2 &= -\frac{1}{E}i\partial_x\chi_1 = -\frac{1}{E}i\partial_x e^{ikx} = \frac{k}{E}e^{ikx} = \pm e^{ikx}. \end{aligned} \tag{5}$$

Hence the solution to the equation is the plane wave
$$\chi = \begin{pmatrix} e^{ikx-iEt} \\ \pm e^{ikx-iEt} \end{pmatrix}.$$

The problem on tunneling a massless particle through a rectangular potential barrier is reduced to the standard equation
$$\hat{H}\psi = E\psi, \tag{6}$$

where
$$\hat{H} = c\hat{p} + \hat{U}, \tag{7}$$

$$\hat{U} = \begin{cases} 0, & x < 0, \\ U_0, & 0 \le x \le a, \\ 0, & x > a. \end{cases} \tag{8}$$

Here c is light velocity, $\hat{p}$ is momentum operator. In this case the dispersion relation is $k_0 = c|k|$, $k = k_x$ is momentum.

3. Tunneling of wave packets

To investigate tunneling of a wave packet having an arbitrary form φ and momentum k we consider a superposition of waves with the coefficients which are determined by the Gauss distribution function

$$\psi(x,t) = \frac{1}{\Delta k\sqrt{2\pi}}\int_0^\infty e^{-\frac{(k-k_0)^2}{2\Delta k^2}}\varphi(k,x,t)dk, \tag{9}$$

where $\varphi(k,x,t) = \varphi(k,x)\cdot e^{-i\frac{E}{\hbar}t}$ in corresponding space intervals reads

$$\begin{aligned} \varphi_I &= e^{ikx} + B_1 e^{-ikx}, \quad x \le 0, \\ \varphi_{II} &= A_2 e^{i\lambda x} + B_2 e^{-i\lambda x}, \quad 0 \le x \le a, \\ \varphi_{\mathrm{III}} &= A_3 e^{ikx}, \quad x \ge a. \end{aligned} \tag{10}$$

Here $\lambda = \frac{E-U}{c\hbar}$, that is a real number always. By using the match conditions

$$\begin{aligned} \varphi_I(0) &= \varphi_{II}(0), \\ \varphi_I'(0) &= \varphi_{II}'(0), \\ \varphi_{II}(a) &= \varphi_{\mathrm{III}}(a), \\ \varphi_{II}'(a) &= \varphi_{III}'(a), \end{aligned} \tag{11}$$

we find the transmitted wave amplitude

$$A_3 = \frac{-4k\lambda}{2\left(k^2+\lambda^2\right)e^{ika}\sinh(i\lambda a) - 4k\lambda e^{ika}\cosh(i\lambda a)}. \tag{12}$$

For a given real λ it can be presented in the exponential form as

$$A_3 = \frac{2k\lambda \exp\left\{i\left(\operatorname{arctg}\left(\frac{k^2+\lambda^2}{2\lambda k}\operatorname{tg}(\lambda a)\right) - ka\right)\right\}}{\sqrt{\sin^2\lambda a \cdot \left(k^2+\lambda^2\right)^2 + 4k^2\lambda^2\cos^2(\lambda a)}}. \tag{13}$$

Thus, the transmitted wave packet looks as follows

$$\psi_{\mathrm{III}}\left(x,t\right) = \frac{1}{\Delta k\sqrt{2\pi}}\int_0^\infty \left|A_3\right| e^{-\frac{(k-k_0)^2}{2\Delta k^2}+i\left(kx-\frac{E}{\hbar}t+\operatorname{arctg}\left(\frac{k^2+\lambda^2}{2\lambda k}\operatorname{tg}(\lambda a)\right)-ka\right)}dk. \tag{14}$$

For the ingoing packet we have

$$\psi_I(x,t) = \frac{1}{\Delta k\sqrt{2\pi}}\int_0^\infty e^{-\frac{(k-k_0)^2}{2\Delta k^2}+i\left(kx-\frac{E}{\hbar}t\right)}dk. \tag{15}$$

Now, let us consider a part of the arbitrary wave packet at the point k:

$$\delta\psi = \int_{k-\delta k}^{k+\delta k} F\left(k'\right)e^{i\varphi(k')}dk'$$

$$\delta\psi = \int\limits_{k-\delta k}^{k+\delta k}\left(F(k) + \left.\frac{\partial F}{\partial k'}\right|_{k'=k}(k'-k) + \ldots\right)e^{i\left(\varphi(k)+\left.\frac{\partial\varphi}{\partial k'}\right|_{k'=k}(k'-k)+\ldots\right)}dk' \approx$$

$$\approx F(k)e^{i\varphi(k)}\int\limits_{k-\delta k}^{k+\delta k} e^{i\left.\frac{\partial\varphi}{\partial k'}\right|_{k'=k}(k'-k)}dk' = 2F(k)e^{i\varphi(k)}\frac{\sin\left(\left.\frac{\partial\varphi}{\partial k'}\right|_{k'=k}\delta k\right)}{\left.\frac{\partial\varphi}{\partial k'}\right|_{k'=k}}$$

The main contributions come from the components satisfying the condition $\left.\frac{\partial\varphi}{\partial k'}\right|_{k'=k} = 0.$

Let us consider from this point of view the ingoing and outgoing packets.

Ingoing packet:

$$\frac{\partial}{\partial k}\left(kx - \frac{Et}{\hbar}\right) = 0, \tag{16}$$

$x = ct$ - free motion. In the point $X = 0$ the particle will appears at the moment $t_0 = 0$.

Outgoing packet:

$$\frac{\partial}{\partial k}\left(kx - \frac{Et}{\hbar} + \operatorname{arctg}\left(\frac{k^2+\lambda^2}{2\lambda k}\operatorname{tg}(\lambda a)\right) - ka\right) = 0. \tag{17}$$

Calculating the derivative we take into account that $\lambda = k - \frac{U}{c\hbar}$. As it is shown in Appendix, this results in the equality:

$$x - ct + \frac{2a\lambda k\left(k^2+\lambda^2\right) - (k+\lambda)(k-\lambda)^2 \sin 2\lambda a}{4k^2\lambda^2\cos^2\lambda a + \left(k^2+\lambda^2\right)^2\sin^2\lambda a} - a = 0 \tag{18}$$

and particle leaves the point $x = a$ of the barrier at the moment t_a,

$$t_a = \frac{1}{c}\frac{2a\lambda k\left(k^2+\lambda^2\right) - (k+\lambda)(k-\lambda)^2 \sin 2\lambda a}{4k^2\lambda^2\cos^2\lambda a + \left(k^2+\lambda^2\right)^2\sin^2\lambda a}. \tag{19}$$

The tunneling time is defined as $\Delta t = t_a - t_0 = t_a$.

Extending the width of the barrier to infinity, we obtain that the Hartman effect does not happen,

$$\lim_{a\to+\infty}\Delta t = \lim_{a\to+\infty}\frac{1}{c}\frac{2a\lambda k\left(k^2+\lambda^2\right) - (k+\lambda)(k-\lambda)^2 \sin 2\lambda a}{4k^2\lambda^2\cos^2\lambda a + \left(k^2+\lambda^2\right)^2\sin^2\lambda a} = \infty.$$

This is in contrast to the known nonrelativistic case [2]. As it is known, in this case the speed of the transition could exceeds the velocity c of the light for some cases of the packets.

4. Conclusion

To summarize our results we note that the case of considered Weyl spinor in an essential way differs from the nonrelativistic situation. In the latter, for not large energy of ingoing particle the energy under the barrier becomes imaginary and this strongly modifies the transmitted amplitude. On the contrary, in case of massless particle the energy is always real. So that the peculiarities proper to the former case absent and the coefficient of transition is given by (23). Hence it becomes clear that the results of paper [2] are the consequences of nonrelativistic dispersion relations used their. From obtained here results, we could conclude that the relativistic wave packets, massless or massive, are not well studied in the problem of the packet transition. That requires new detailed investigations. First of all this concerns relativistic particles with more general dispersion relations, possibly, in various external conditions. For example, external magnetic field, where charged particles freely move along the field direction. So, the obtained results could be useful. This is the problem left for the future.

A. Appendix

In this appendix we find derivatives used in calculating (17)- (19). Computing the necessary derivatives we obtain

$$\begin{aligned} \lambda &= k - \frac{U}{c\hbar}, \lambda'_k = 1 \qquad (20)\\ \frac{\partial}{\partial k}\left(\frac{k^2+\lambda^2}{2\lambda k}\right) &= \frac{2(k+\lambda)\cdot 2\lambda k - \left(k^2+\lambda^2\right)\cdot 2(k+\lambda)}{4k^2\lambda^2} \\ &= -\frac{2(k+\lambda)(k-\lambda)^2}{4k^2\lambda^2}. \end{aligned}$$

$$\frac{\partial}{\partial k}\left(\frac{k^2+\lambda^2}{2\lambda k}\operatorname{tg}\lambda a\right) = \frac{2a\lambda k\left(k^2+\lambda^2\right) - (k+\lambda)(k-\lambda)^2 \sin 2\lambda a}{4k^2\lambda^2\cos^2\lambda a} \tag{21}$$

$$\begin{aligned}
&\frac{\partial}{\partial k}\,\mathrm{arctg}\left(\frac{k^2+\lambda^2}{2\lambda k}\,\mathrm{tg}(\lambda a)\right) = \\
= \quad &\frac{2a\lambda k\left(k^2+\lambda^2\right) - (k+\lambda)(k-\lambda)^2\sin 2\lambda a}{4k^2\lambda^2\cos^2\lambda a + \left(k^2+\lambda^2\right)^2\sin^2\lambda a}
\end{aligned} \tag{22}$$

$$\begin{aligned}
|A_3|^2 \quad &= \quad \frac{2ik\lambda e^{-ika}}{(k^2+\lambda^2)\sin\lambda a + 2ik\lambda\cos\lambda a}\cdot\frac{-2ik\lambda e^{ika}}{(k^2+\lambda^2)\sin\lambda a - 2ik\lambda\cos\lambda a} \\
&= \quad \frac{4k^2\lambda^2}{\left((k^2+\lambda^2)\sin\lambda a\right)^2 + (2k\lambda\cos\lambda a)^2}.
\end{aligned} \tag{23}$$

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