DYNAMICS OF A SYSTEM OF TWO-LEVEL EMITTERS IN THE DICKE MODEL TAKING INTO ACCOUNT SMALL CORRELATIONS BETWEEN THEM

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On the basis of our previous study of the quantum system of atoms and the electromagnetic field, a system of non-relativistic stationary and non-interacting small one-electron atoms located in a region much smaller in size than the characteristic wavelength of the field is considered. Atoms are considered in the two-level approximation and interact with the field by dipole-electric interaction. We carefully analyze the state space of the atom and the operators in it in the spin $s = 1/2$ formalism, which leads to the Dicke Hamiltonian in terms of the spin formalism.

Non-equilibrium states of the system are investigated using the reduced description method and described by the occupation numbers of photon states $n_{\alpha k}$, the degree of excitation of the atoms η_1, and the value of correlations between them η_2. The basis of our consideration is the Peletminsky–Yatsenko model, the foundations of which and, in particular, the approach to solving the Cauchy problem, based on the notion of effective initial conditions, are thoroughly discussed.

Since the Dicke model deals with the dynamics of a system of particles with spin, the technique of calculating the average values of products of spin operator proposed by Vax, Larkin, and Pikin is developed to a certain extent. Averages in a state with a statistical operator of a system of atoms with given excitations are considered. The obtained results are maximally simplified in the case of spin $s = 1/2$.

The impossibility of calculations with a quasi-local statistical operator when describing the state of the system with parameters $n_{\alpha k}$, η_1, η_2 is overcome with using a somewhat simplified approach of our previous work, in which, instead of a parameter η_2, the system is described by a small deviation $\delta\eta_2 = \eta_2 - \eta_{20}$ of the parameter η_2 from its value η_{20} when describing the system only by the parameters $n_{\alpha k}$, η_1. Developed approaches to calculations with spin operators are used to calculate the quasi-equilibrium statistical operator when choosing of the quantities $n_{\alpha k}$, η_1, and $\delta\eta_2$ as reduced description parameters and the right-hand sides of the time equations for these parameters.

Keywords: superradiance, Dicke model, reduced description method, calculation with spin operators, small correlation dynamics, photon non-equilibrium states, long wave limit.

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1. Introduction

Modern physics deals with very fine phenomena, measurements, and tools for obtaining and processing the signals concerning the researched problems. In this view the correlation properties of emissions of different nature are of interest. Lasers based on stimulated emission of multiparticle systems of emitters are usual instruments of physical investigations now. Along with them, since the pioneer paper by Dicke [1], the superradiance phenomenon attracts serious attention of physicists. The Dicke process essence is the cooperative spontaneous emission. The development of correlations in a system without resonator opens the way to the coherent emission generation in new frequency ranges and gives a unique example of self-organization in non-equilibrium processes [2]. The consistent description of Dicke superradiance needs quantum notions, and the new optics section – quantum optics is formulated in terms of correlation functions [3]. Just in such way is constructed the theory of Dicke superradiance [4]. The peculiarity of this phenomenon description is the transition to the dynamics of emitter subsystem in accordance with Bogolyubov's idea of "fast" and "slow" processes [5]. Herewith the most demonstrative parameter (level occupancy) remains almost constant for a long time and the correlation characteristics change exponentially [6].

The general picture, especially in a prolonged emitter system, is still hard for study. The authors started the investigations of Dicke model in the early 2000-ies and were the first who applied the Bogolyubov reduced description method (RDM) to this problem. Our interests touched the questions about the influence of the system geometry, own motion of emitters, correlation properties of the generated field [7-9]. RDM provides the possibility of solving the problem of correlator decoupling in the method of boson variable elimination [5, 6] but faces the problem of calculating averages with the spin Hamiltonian of complex structure. This paper presents the new results concerning the indicated difficulties.

The work has the following structure. Section 2 presents the substantiation of the Dicke model in detail. Section 3 describes basics of the atom and electromagnetic field dynamics in the reduced description method. Section 4 presents some new ideas in spin operator calculations. Section 5 is devoted to small correlation dynamics of the Dicke model. Section 6 realizes emitter-field dynamics with the spin operator technique.

2. The Dicke model of a superradiant system

A system of atoms with one electron, which are stationary in space and interact with a system of photons, is considered. The Hamiltonian operator of such a system in the main approximation (quasi-relativistic theory, small size of atoms) has the form [10]

$$\hat{H} = \sum_{\alpha,\vec{k}} \hbar\omega_k c^+_{\alpha\vec{k}} c_{\alpha\vec{k}} + \sum_n \hat{H}_{ne} - \sum_n \hat{\vec{d}}_n \hat{\vec{E}}(\vec{x}_n)\,,$$
$$\hat{H}_{ne} = \frac{\hat{\vec{p}}^{\,2}_{ne}}{2m} - \frac{e^2}{\hat{\vec{x}}^{\,2}_{ne}}\,, \qquad \hat{\vec{d}}_n = -e\hat{\vec{x}}_{ne} \qquad (\sum_n ... \equiv \sum_{1\le n\le N} ...). \tag{2.1}$$

Here $\vec{x}_n$ is a radius-vector of the atom nucleus, $\vec{x}_{ne}$ is a relative radius-vector of the electron and $\vec{p}_{ne}$ is a corresponding momentum, m, e are electron mass and charge modulus, $\hat{\vec{d}}_n$ is atom dipole moment, $\hat{\vec{E}}(\vec{x})$ is a transversal electric field operator

$$\hat{\vec{E}}(\vec{x}) = \sum_{\alpha,\vec{k}} \left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} \left(\vec{e}_{\alpha\vec{k}} c_{\alpha\vec{k}} e^{i\vec{k}\vec{x}} + \vec{e}^{\,*}_{\alpha\vec{k}} c^+_{\alpha\vec{k}} e^{-i\vec{k}\vec{x}}\right) \tag{2.2}$$

in standard designations of quantum electrodynamics. Particularly $c^+_{\alpha\vec{k}}$, $c_{\alpha\vec{k}}$ are operators of creation and annihilation of a photon in the state with polarization α and wave vector $\vec{k}$ (these operators traditionally are written without hats), $\vec{e}_{\alpha\vec{k}}$ are polarization vectors (they can be chosen real), $\omega_k = ck$ is photon spectrum. We use periodical boundary conditions and suppose taking the limit $V \to \infty$ after the calculations are complete. At the same time the atom system is considered small compared to the wavelength of the photon field λ

$$|\vec{x}_n| \le L \quad (1 \le n \le N); \qquad L << \lambda \quad (\lambda = 2\pi / k). \tag{2.3}$$

In such case, we may choose the electric field operator independent of coordinates in our research

$$\hat{\vec{E}}(\vec{x}) = \sum_{\alpha,\vec{k}} \left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2} \vec{e}_{\alpha\vec{k}} (c_{\alpha\vec{k}} + c^+_{\alpha\vec{k}})\,. \tag{2.4}$$

In the two-level approximation only electron transitions between states $|n,\alpha\rangle$ $(\alpha = 1,2)$ in each atom are taken into account

$$\hat{H}_{ne}\,|\,n,\alpha\rangle = E_\alpha\,|\,n,\alpha\rangle\,,\quad \langle n,\alpha\,|\,n,\alpha'\rangle = \delta_{\alpha\alpha'}\,,\quad \sum_{\alpha=1,2}|\,n,\alpha\rangle\langle n,\alpha\,| = \hat{1}_n\,. \tag{2.5}$$

The basis $|\,n,\alpha\rangle$ can be chosen in such way that atom dipole moment operator $\hat{\vec{d}}_n$ has the matrix elements in it [11]

$$\langle n,\alpha\,|\,\hat{\vec{d}}_n\,|\,n,\alpha\rangle = 0\,,\qquad \langle n,1|\,\hat{\vec{d}}_n\,|\,n,2\rangle = \langle n,2\,|\,\hat{\vec{d}}_n\,|\,n,1\rangle = \vec{e}_n d\,, \tag{2.6}$$

where d is a positive scalar, which is the same for all atoms, and $\vec{e}_n$ is a unit vector with own direction for each atom. The first equality follows from the fact that the operator $\hat{\vec{d}}_n$ is a polar vector, and the second one resembles an arbitrary phase factor in the eigenvector definition.

Formulas (2.5) and (2.6) allow to write the dipole moment operator $\hat{\vec{d}}_n$ and the Hamiltonian $\hat{H}_{ne}$ in the form

$$\begin{gathered}
\hat{\vec{d}}_n = \sum_{\alpha,\alpha'=1,2}|\,n,\alpha\rangle\langle n,\alpha\,|\hat{\vec{d}}_n\,|\,n,\alpha'\rangle\langle n,\alpha'\,| = (|\,n,1\rangle\langle n,2\,| + |\,n,2\rangle\langle n,1\,|)\vec{e}_n d = \\
= \vec{e}_n d\sum_{\alpha,\alpha'=1,2}|\,n,\alpha\rangle\langle n,\alpha'\,|\sigma^x_{\alpha\alpha'} \\
\hat{H}_{ne} = \sum_{\alpha=1,2}|\,n,\alpha\rangle\langle n,\alpha\,|\,E_\alpha = |\,n,1\rangle\langle n,1\,|\,E_1 + |\,n,2\rangle\langle n,2\,|\,E_2 = \\
= (|\,n,1\rangle\langle n,1| - |\,n,2\rangle\langle n,2\,|)(E_1 - E_2)/2 + (|\,n,1\rangle\langle n,1| + |\,n,2\rangle\langle n,2\,|)(E_1 + E_2)/2 = \\
= (E_1 - E_2)/2\sum_{\alpha,\alpha'=1,2}|\,n,\alpha\rangle\langle n,\alpha'\,|\sigma^z_{\alpha\alpha'} + \hat{1}(E_1 + E_2)/2
\end{gathered} \tag{2.7}$$

Here we use Pauli matrices

$$\sigma^x_{\alpha\alpha'} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\qquad \sigma^y_{\alpha\alpha'} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix},\qquad \sigma^z_{\alpha\alpha'} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{2.8}$$

with doubled matrix elements of the operators

$$\begin{gathered}
\hat{s}^x_n = \sum_{\alpha,\alpha'=1,2}|\,n,\alpha\rangle\langle n,\alpha'\,|\sigma^x_{\alpha\alpha'}/2\,,\qquad \hat{s}^y_n = \sum_{\alpha,\alpha'=1,2}|\,n,\alpha\rangle\langle n,\alpha'\,|\sigma^y_{\alpha\alpha'}/2\,, \\
\hat{s}^z_n = \sum_{\alpha,\alpha'=1,2}|\,n,\alpha\rangle\langle n,\alpha'\,|\sigma^z_{\alpha\alpha'}/2\,.
\end{gathered} \tag{2.9}$$

In these terms the dipole moment operator $\hat{\vec{d}}_n$ and the Hamiltonian $\hat{H}_{ne}$ take the form

$$\hat{\vec{d}}_n = 2d\,\vec{e}_n s^x_n\,,\qquad \hat{H}_{ne} = \hbar\omega_0\hat{s}^z_n\qquad (\hbar\omega_0 \equiv E_1 - E_2\,). \tag{2.10}$$

The last term in the atom Hamiltonian in (2.7) determines only the beginning of the energy countdown, so it is omitted here. As a result, the Hamiltonian of the system of atoms and photons (2.1) is given by the formula

$$\hat{H} = \hat{H}_0 + \hat{H}_1,$$
$$\hat{H}_0 = \sum_{\alpha,\vec{k}} \hbar\omega_k c^+_{\alpha\vec{k}} c_{\alpha\vec{k}} + \sum_n \hbar\omega_0 \hat{s}^z_n , \qquad \hat{H}_1 = \sum_{n,\alpha,\vec{k}} \lambda_{n\alpha\vec{k}} (c_{\alpha\vec{k}} + c^+_{\alpha\vec{k}})(\hat{s}^+_n + \hat{s}^-_n) , \tag{2.11}$$
$$\lambda_{n\alpha\vec{k}} \equiv \left(\frac{2\pi\hbar}{V}\right)^{1/2} \vec{e}_{\alpha\vec{k}} \vec{e}_n d \qquad (\hat{s}^\pm_n \equiv \hat{s}^x_n \pm i\hat{s}^y_n),$$

which can be named the Dicke Hamiltonian.

It is easy to check that the commutation relations

$$[\hat{s}^x_n, \hat{s}^y_{n'}] = i\hat{s}^z_n \delta_{nn'} , \qquad [\hat{s}^x_n, \hat{s}^z_{n'}] = -i\hat{s}^y_n \delta_{nn'} , \qquad [\hat{s}^y_n, \hat{s}^z_{n'}] = i\hat{s}^x_n \delta_{nn'} \tag{2.12}$$

are fulfilled for operators from (2.9), so they have the properties of spin operators (without the factor $\hbar$). That is why the notation $\hat{s}^x_n, \hat{s}^y_n, \hat{s}^z_n$ is used for them in this paper. In [10] these operators were named Dicke operators and designated $\hat{\mathbf{r}}_{nx}, \hat{\mathbf{r}}_{ny}, \hat{\mathbf{r}}_{nz}$ in accordance with Dicke [1], but the successive derivation of the Hamiltonian (2.11) (see (2.5) – (2.10)) is absent in the fundamental work.

3. Cauchy problem in the reduced description method for nonequilibrium processes

The non-equilibrium states of the system are described by the average values $\mathrm{Sp}\,\rho(t)\hat{\eta}_a$ of some parameters, where $\hat{\eta}_a$ are the operators of these parameters (a is the parameter number). In the Dicke model described in the previous section, these are the parameters η_a: $\eta_1, \eta_2, \eta_{\alpha\vec{k}} \equiv n_{\alpha\vec{k}}$, which correspond to the operators $\hat{\eta}_a$.

$$\hat{\eta}_1 = \hat{s}^z , \qquad \hat{\eta}_2 = \hat{s}^+\hat{s}^- , \qquad \hat{\eta}_{\alpha\vec{k}} = \hat{n}_{\alpha\vec{k}} \equiv c^+_{\alpha\vec{k}} c_{\alpha\vec{k}} ,$$
$$\hat{s}^z \equiv \sum_{1\le n\le N} \hat{s}^z_n , \qquad \hat{s}^\pm \equiv \sum_{1\le n\le N} \hat{s}^\pm_n . \tag{3.1}$$

In this problem, the value η_1 determines the degree of excitation of the system of atoms, and η_2 characterizes the correlations in it; $n_{\alpha\vec{k}}$ is the average number of photons in the state $\alpha\vec{k}$. In the literature [5, 12], it is believed that these parameters adequately describe the phenomenon of superradiance.

Our study of non-equilibrium states of the Dicke model is based on Bogolyubov's reduced description method [13]. It is based on the functional hypothesis, according to which the statistical operator (SO) of a non-equilibrium system $\rho(t)$ at large times $t >> \tau_0$ depends on time and the initial state of the system $\rho(t=0) \equiv \rho_0$ only through the mediation of a limited number of parameters, which are called reduced description parameters (RDPs).

$$\rho(t) \xrightarrow[t>>\tau_0]{} \rho(\eta(t,\rho_0)) \qquad (\mathrm{Sp}\,\rho(\eta) = 1, \quad \mathrm{Sp}\,\rho(\eta)\hat{\eta}_a = \eta_a) \tag{3.2}$$

і визначаються формулою

$$\mathrm{Sp}\,\rho(t)\hat{\eta}_a \xrightarrow[t>>\tau_0]{} \eta_a(t,\rho_0) . \tag{3.3}$$

($\hat{\eta}_a$ are RDP operators). The arrow in formulas (3.2) and (3.3) indicates that their right parts are the asymptotics of the left part. It is convenient to write the solution of the quantum equation using the Liouville operator $\mathbf{L}$

$$\partial_t \rho(t) = \mathbf{L}\rho(t), \qquad \mathbf{L}\rho \equiv \frac{i}{\hbar}[\rho, \hat{H}]; \qquad \rho(t) = e^{t\mathbf{L}}\rho_0. \tag{3.4}$$

The basic ideas of RDM are important for our research. The leading statement is that the statistical operator $\rho(\eta(t,\rho_0))$ exactly satisfies the Liouville equation

$$\partial_t \rho(\eta(t,\rho_0)) = \frac{i}{\hbar}[\rho(\eta(t,\rho_0)), \hat{H}] \tag{3.5}$$

with the Hamilton operator $\hat{H}$, and the parameters $\eta_a(t,\rho_0)$ exactly satisfy the time equation

$$\partial_t \eta_a(t,\rho_0) = L_a(\eta(t,\rho_0)), \qquad L_a(\eta) \equiv \frac{i}{\hbar}\mathrm{Sp}\,\rho(\eta)[\hat{H}, \hat{\eta}_a]. \tag{3.6}$$

In RDM applications, in particular for studying the states of the Dicke model, it is important to study the influence of the initial state of the system on its evolution [5, 12]. To consider the Cauchy problem, the solutions of equations (3.5) and (3.6) are continued to $t = 0$, although they describe the system evolution only for $t >> \tau_0$. Herewith, the values of the functions $\eta_a(t,\rho_0)$ at $t = 0$ are called effective initial conditions.

Taking into account the expression for the function $L_a(\eta)$, the SO $\rho(\eta)$ satisfies the nonlinear differential equation

$$\sum_a \frac{\partial \rho(\eta)}{\partial \eta_a} L_a(\eta) = \frac{i}{\hbar}[\rho(\eta), \hat{H}], \tag{3.7}$$

which can be solved only approximately in some perturbation theory. The case of the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ with the main $\hat{H}_0$ and small $\hat{H}_1$ parts is particularly important. Formally, $\hat{H} \sim \lambda^0$, $\hat{H}_1 \sim \lambda^1$ where λ is a small parameter).

Consideration of non-equilibrium states of the system begins with the selection of RDPs and their operators $\hat{\eta}_a$. The previous development of the theory of non-equilibrium processes is the basis for this work. At the same time, the modern trend is to expand the set of RDPs by taking into account non-equilibrium correlations (fluctuations) of the studied RDPs.

A constructive approach to the selection of RDPs was proposed by Peletminsky and Yatsenko using the symmetries of the basic Hamiltonian $\hat{H}_0$. This led them (see the original paper [14], as well as [13]) to a model (Peletminsky–Yatsenko model, in further PYa model) in which the RDPs operators satisfy the condition

$$[\hat{H}_0, \hat{\eta}_a] = \sum_b c_{ab}\hat{\eta}_b, \tag{3.8}$$

where c_{ab} is a C-number matrix. For the study of superradiance in the Dicke model, we proposed RDPs with operators (3.1) while operator $\hat{H}_0$ (2.11) is the main contribution to the Hamiltonian. Since simple ratios are valid

$$[\hat{H}_0, \hat{\eta}_a] = 0, \qquad [\hat{\eta}_a, \hat{\eta}_b] = 0, \tag{3.9}$$

non-equilibrium processes in the Dicke model can be investigated within the framework of the PYa model. In this regard, we will discuss the important results of this model theory. Based on it, the authors established a functional hypothesis in the basic approximation of perturbation theory [13]

$$e^{t\mathbf{L}_0}\rho_0 \xrightarrow[t>>\tau_0]{} \rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}t}\mathrm{Sp}\rho_0\hat{\eta})) \qquad \left(\mathbf{L}_0\rho \equiv \frac{i}{\hbar}[\rho,\hat{H}_0]\right). \tag{3.10}$$

Such formulation includes a SO

$$\rho_q(Y) = \exp\left\{\Omega(Y) - \sum\nolimits_a Y_a\hat{\eta}_a\right\}, \qquad \mathrm{Sp}\,\rho_q(Y) \equiv 1 \tag{3.11}$$

which is called quasi-equilibrium (it is close to the equilibrium SO in certain cases). Functions $Z_a(\eta)$ in (2.7) are determined by the condition

$$\mathrm{Sp}\rho_q(Z(\eta))\hat{\eta}_a = \eta_a\,. \tag{3.12}$$

Bogolyubov showed that equation (2.7) for the statistical operator $\rho(\eta)$ is invariant with respect to time inversion and should be supplemented with a boundary condition that selects the physical direction of time [15]. As such a condition, the authors of the model chose the functional hypothesis (3.10) in the main approximation of the perturbation theory for the SO $\rho_0 = \rho(\eta)$, which, taking into account (3.2), takes the form

$$e^{t\mathbf{L}_0}\rho(\eta) \xrightarrow[t>>\tau_0]{} \rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}t}\eta) \tag{3.13}$$

and is written in terms of evolution in the physical direction of time. This relation made it possible to obtain a nonlinear integral equation from the nonlinear differential equation (3.7)

$$\rho(\eta) = \rho_q(Z(\eta)) + \int_{-\infty}^{0} d\tau e^{-\tau\mathbf{L}_0}\mathrm{f}\,(e^{\frac{i}{\hbar}c\tau}\eta)\,, \tag{3.14}$$

where denoted

$$\mathrm{f}\,(\eta) = \frac{i}{\hbar}\left[\,\rho(\eta),\hat{H}_1\right] - \sum\nolimits_a \frac{\partial\rho(\eta)}{\partial\eta_a}M_a(\eta)\,, \qquad M_a(\eta) = \frac{i}{\hbar}\mathrm{Sp}\,\rho(\eta)[\hat{H}_1,\hat{\eta}_a]\,. \tag{3.15}$$

In these terms the right-hand side of the time equation (3.6) for RDP $\eta_a(t,\rho_0)$ with considering (3.8) takes the form

$$L_a(\eta) = \frac{i}{\hbar}\sum_b c_{ab}\eta_b + M_a(\eta) \tag{3.16}$$

Equation (3.14) can be solved by an iterative procedure in perturbation theory by $\hat{H}_1$ because its integrand expression has the first order in this operator. For the SO $\rho(\eta)$ we have

$$\rho(\eta) = \sum_{n=0}^{\infty}\rho^{(n)}(\eta)\,, \qquad \rho^{(0)}(\eta) = \rho_q(Z(\eta))\,,$$
$$\rho^{(1)}(\eta) = \int_{-\infty}^{0} d\tau e^{-\mathbf{L}_0\tau}\left\{\frac{i}{\hbar}\left[\rho_q(Z(\eta)),\hat{H}_1\right] - \sum_a \frac{\partial\rho_q(Z(\eta))}{\partial\eta_a}M_a^{(1)}(\eta)\right\}_{\eta\to e^{\frac{i}{\hbar}c\tau}\eta}\,. \tag{3.17}$$

The right-hand side $L_a(\eta)$ of the time equation for RDP is given by the relation (3.16) and formulas

$$M_a(\eta)=\sum_{n=1}^{\infty}M_a^{(n)}(\eta)\,,\qquad M_a^{(1)}(\eta)=\frac{i}{\hbar}\mathrm{Sp}\rho_q(Z(\eta))\left[\hat{H}_1,\hat{\eta}_a\right],$$
$$M_a^{(2)}(\eta)=\frac{i}{\hbar}\mathrm{Sp}\rho^{(1)}(\eta)\left[\hat{H}_1,\hat{\eta}_a\right]. \tag{3.18}$$

Let's move on to the calculation of the effective conditions, somewhat simplifying the approach of [13]. From (3.4), (3.8) we consistently have

$$\partial_t\,\mathrm{Sp}\rho(t)\hat{\eta}_a=\frac{i}{\hbar}\mathrm{Sp}\rho(t)[\hat{H},\hat{\eta}_a]=\frac{i}{\hbar}\sum_b c_{ab}\,\mathrm{Sp}\rho(t)\hat{\eta}_b+\frac{i}{\hbar}\mathrm{Sp}\rho(t)[\hat{H}_1,\hat{\eta}_a]\,,$$
$$\partial_t\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\,\mathrm{Sp}\rho(t)\hat{\eta}_b=\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\,\frac{i}{\hbar}\mathrm{Sp}\rho(t)[\hat{H}_1,\hat{\eta}_b]\,, \tag{3.19}$$
$$\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\,\mathrm{Sp}\rho(t)\hat{\eta}_b-\mathrm{Sp}\rho_0\hat{\eta}_a=\sum_b\int_0^t d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\,\frac{i}{\hbar}\mathrm{Sp}\rho(\tau)[\hat{H}_1,\hat{\eta}_b]$$

According to (3.4) and (3.5), $\rho(t)$ and $\rho(\eta(t,\rho_0))$ satisfy the same time equation and therefore, taking into account (3.2), the following formulas are valid

$$\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\,\mathrm{Sp}\rho(\eta(t,\rho_0))\hat{\eta}_b-\mathrm{Sp}\rho(\eta(0,\rho_0))\hat{\eta}_a=$$
$$=\sum_b\int_0^t d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\,\frac{i}{\hbar}\mathrm{Sp}\rho(\eta(\tau,\rho_0))[\hat{H}_1,\hat{\eta}_b]\,, \tag{3.20}$$
$$\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\,\eta_b(t,\rho_0)-\eta_a(0,\rho_0)=\sum_b\int_0^t d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\,\frac{i}{\hbar}\mathrm{Sp}\rho(\eta(\tau,\rho_0))[\hat{H}_1,\hat{\eta}_b]\,.$$

The last formula is analogous to the previous one. Their difference has the form

$$\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\,\mathrm{Sp}\rho(t)\hat{\eta}_b-\mathrm{Sp}\rho_0\hat{\eta}_a-\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\,\eta_b(t,\rho_0)+\eta_a(0,\rho_0)=$$
$$=\sum_b\int_0^t d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\,\frac{i}{\hbar}\mathrm{Sp}\{\rho(\tau)-\rho(\eta(\tau,\rho_0))\}[\hat{H}_1,\hat{\eta}_b]\,. \tag{3.21}$$

In this relation, considering the functional hypothesis and the definition of the function $\eta_a(t,\rho_0)$, it is possible to go to the limit $t\to+\infty$, which gives

$$\eta_a(0,\rho_0)-\mathrm{Sp}\rho_0\hat{\eta}_a=$$
$$=\sum_b\int_0^{+\infty} d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\,\frac{i}{\hbar}\mathrm{Sp}\{\rho(\tau)-\rho(\eta(\tau,\rho_0))\}[\hat{H}_1,\hat{\eta}_b]\,. \tag{3.22}$$

Taking into account formula (3.4), its analogue for $\rho(\eta(\tau,\rho_0))$, and the identity $\mathrm{Sp}(e^{t\mathbf{L}}\hat{a})\hat{b}=\mathrm{Sp}\,\hat{a}e^{-t\mathbf{L}}\hat{b}$, we obtain the integral equation for $\eta_a(0,\rho_0)$

$$\eta_a(0,\rho_0)=\mathrm{Sp}\rho_0\hat{\eta}_a+\sum_b\int_0^{+\infty} d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\,\frac{i}{\hbar}\mathrm{Sp}\{\rho_0-\rho(\eta(0,\rho_0))\}e^{-\tau\mathbf{L}}[\hat{H}_1,\hat{\eta}_b]\,. \tag{3.23}$$

This integral equation is solved in the perturbation theory in interaction $\hat{H}_1$, which, in accordance with (3.8), gives

$$\eta_a(0,\rho_0) = \mathrm{Sp}\,\rho_0\hat{\eta}_a + \\ + \int_0^{+\infty} d\tau \frac{i}{\hbar} \mathrm{Sp}\{\rho_0 - \rho_q(Z(\mathrm{Sp}\,\rho_0\hat{\eta}))\}[\hat{H}_1(\tau),\hat{\eta}_b] + O(\lambda^2) \qquad \left(\hat{H}_1(t) \equiv e^{-t\mathbf{L}_0}\hat{H}_1\right). \tag{3.24}$$

This formula makes it possible to investigate the Cauchy problem in the Peleminsky–Yatsenko model for an arbitrary initial state of the system.

4. The technique of calculating averages with spin operators

The key to the implementation of the reduced description method in the study of non-equilibrium states of the Dicke model in the technique of calculations with spin operators. We will follow the mathematical developments [16, 17] for arbitrary spin s and calculations with a statistic operator that describes a system of atoms with given excitations. This is designed to consider the Dicke model and its generalizations.

In fact, in this section we will discuss the calculation of the average values of products of spin operators $\hat{\vec{s}}_n = (\hat{s}_n^x, \hat{s}_n^y, \hat{s}_n^z)$ $(1 \le n \le N)$ for a system of N identical particles with spin s and the usual commutation relations

$$[\hat{s}_n^x, \hat{s}_{n'}^y] = i\hat{s}_n^z\delta_{nn'}, \qquad [\hat{s}_n^x, \hat{s}_{n'}^z] = -i\hat{s}_n^y\delta_{nn'}, \qquad [\hat{s}_n^y, \hat{s}_{n'}^z] = i\hat{s}_n^x\delta_{nn'}. \tag{4.1}$$

Particles can be considered stationary and located in some points of space x_n. For our purposes, it is convenient not to include Planck's constant in these operators. Along with them, such their combinations are widely used

$$\hat{s}_n^{\pm} = \hat{s}_n^x \pm i\hat{s}_n^y \tag{4.2}$$

with properties

$$[\hat{s}_n^z, \hat{s}_{n'}^{\pm}] = \pm\hat{s}_n^{\pm}\delta_{nn'}, \qquad [\hat{s}_n^+, \hat{s}_{n'}^-] = 2\hat{s}_n^z\delta_{nn'}. \tag{4.3}$$

Particle numbers $n_1, n_2, \ldots$ are further used to abbreviate a record in the form of such type

$$\hat{s}_{n_1}^x = \hat{s}_1^x, \quad \hat{s}_{n_1}^{\pm} = \hat{s}_1^{\pm}, \qquad \delta_{n_1 n_2} = \delta_{12}, \qquad \sum_{n_1=1}^{N} \ldots = \sum_1 \ldots. \tag{4.4}$$

We do the same with other quantities with index n. Herewith the compact formulas are valid

$$[\hat{s}^z, \hat{s}_1^{\pm}] = \pm\hat{s}_1^{\pm}, \quad [\hat{s}_1^z, \hat{s}_2^+\hat{s}_3^-] = \hat{s}_2^+\hat{s}_3^-(\delta_{12} - \delta_{13}), \quad [\hat{s}^z, \hat{s}^+\hat{s}^-] = 0, \quad [\hat{s}^z, \hat{s}_1^+\hat{s}_2^-] = 0 \\ (\sum_1 \hat{s}_1^z \equiv \hat{s}^z, \quad \sum_1 \hat{s}_1^{\pm} \equiv \hat{s}^{\pm}). \tag{4.5}$$

The state space of one particle with spin s has the basis formed by vectors $|\sigma_n, s\rangle$

$$\hat{s}_n^z|\sigma_n, s\rangle = \sigma_n|\sigma_n, s\rangle \qquad (\sigma_n = -s, -s+1, \ldots, s); \\ \hat{\vec{s}}_n^2|\sigma_n, s\rangle = s(s+1)|\sigma_n, s\rangle \qquad \left(\hat{\vec{s}}_n^2 = \hat{s}_n^{x2} + \hat{s}_n^{y2} + \hat{s}_n^{z2} = \hat{s}_n^{z2} + \hat{s}_n^+\hat{s}_n^-\right). \tag{4.6}$$

The state space of a system of N spins is a direct product of N such spaces, and its basis is given by the expression

$$|\sigma\rangle \equiv |\sigma_1, s\rangle ... |\sigma_N, s\rangle \tag{4.7}$$

and the trace of an arbitrary spin operator $\hat{S}$ is defined by the formula

$$\mathrm{Sp}\,\hat{S} = \sum_{\sigma} \langle\sigma|\hat{S}|\sigma\rangle, \qquad \sum\nolimits_{\sigma} ... \equiv \sum\nolimits_{1\le n\le N} \sum\nolimits_{-s\le\sigma_n\le s} ...$$
$$(\mathrm{Sp}\,\hat{S}_1\hat{S}_2 = \mathrm{Sp}\,\hat{S}_2\hat{S}_1). \tag{4.8}$$

In the following, arbitrary products of spin operators $\hat{s}_n^x$, $\hat{s}_n^y$, $\hat{s}_n^z$ of all particles are considered as $\hat{S}$. According to (4.2), we can limit ourselves to products of operators $\hat{s}_n^+$, $\hat{s}_n^-$, $\hat{s}_n^z$, which is more convenient.

In the general case, the state of a system is described with a statistical operator. The SO

$$\rho_0 = e^{\Omega + \sum_{1\le n\le N} h_n \hat{s}_n^z} \qquad (\mathrm{Sp}\,\rho_0 \equiv 1), \tag{4.9}$$

plays an important role. It can be called the SO of a system of N spins in an external field h_n, or a system of particles with known excitations. The normalization condition $\mathrm{Sp}\,\rho_0 \equiv 1$ gives such expressions for the functions Ω and ρ_0 through the value of a statistical sum Z

$$\Omega = -\ln Z, \quad \rho_0 = e^{\sum_{1\le n\le N} h_n \hat{s}_n^z} / Z \qquad Z \equiv \mathrm{Sp}\,e^{\sum_{1\le n\le N} h_n \hat{s}_n^z}. \tag{4.10}$$

According to (4.8), we have

$$Z = \sum\nolimits_{\sigma} \langle\sigma| \prod_{1\le n\le N} e^{h_n \hat{s}_n^z} |\sigma\rangle =$$
$$= \prod_{1\le n\le N} \sum_{-s\le\sigma_n\le s} e^{h_n\sigma_n} = \prod_{1\le n\le N} (e^{h_n(s+1)} - e^{-h_n s}) / (e^{h_n} - 1) = \tag{4.11}$$
$$= \prod_{1\le n\le N} \mathrm{sh} h_n (s + \frac{1}{2}) / \mathrm{sh} h_n \frac{1}{2}.$$

The average value $\langle\hat{S}\rangle$ of an arbitrary spin operator $\hat{S}$ in the state with the CO ρ_0 is given by the expression

$$\langle\hat{S}\rangle = \mathrm{Sp}\,\rho_0 \hat{S}. \tag{4.12}$$

The average value of the product of operators $\hat{s}_n^z$, according to (4.10), is determined by the formula

$$\langle\hat{s}_1^z ... \hat{s}_m^z\rangle = \mathrm{Sp}\,\hat{s}_1^z ... \hat{s}_m^z e^{\sum_{1\le n\le N} h_n \hat{s}_n^z} / Z = \frac{\partial^m Z}{\partial h_1 ... \partial h_m} / Z \tag{4.13}$$

and, particularly,

$$\langle\hat{s}_n^z\rangle = \frac{\partial \ln Z}{\partial h_n} = (s + \frac{1}{2}) \mathrm{cth} h_n (s + \frac{1}{2}) - \frac{1}{2} \mathrm{cth} h_n \frac{1}{2} \equiv b(h_n). \tag{4.14}$$

In the case $s = 1/2$ this formula gives

$$\langle\hat{s}_n^z\rangle = b(h_n) \qquad (s = \frac{1}{2},\ b(h) \equiv \frac{1}{2} \mathrm{th} \frac{h}{2}) \tag{4.15}$$

because $2\mathrm{cth} 2x - \mathrm{cth} x = \mathrm{th} x$.

Formula (4.13) using (4.14) takes the form

$$\langle \hat{s}_1^z ... \hat{s}_m^z \rangle = Z^{-1} \frac{\partial^{m-1}}{\partial h_2 ... \partial h_m} \frac{\partial Z}{\partial h_1} = Z^{-1} \frac{\partial^{m-1}}{\partial h_2 ... \partial h_m} Zb(h_1)$$

and in further is applied for calculating the averages $\langle \hat{s}_1^z ... \hat{s}_m^z \rangle$

$$\langle \hat{s}_1^z ... \hat{s}_m^z \rangle = Z^{-1} \frac{\partial^{m-1}}{\partial h_2 ... \partial h_m} Zb(h_1) \qquad \left(\frac{\partial Z}{\partial h_n} = Zb(h_n), \quad \frac{\partial b(h_1)}{\partial h_m} = \frac{\partial b(h_1)}{\partial h_1} \delta_{1m}\right). \tag{4.16}$$

This result somewhat clarify its proof in [16, 17]. From this formula, in particular, we find

$$\langle \hat{s}_1^z \hat{s}_2^z \rangle = b_1 b_2 + b_1' \delta_{12},$$
$$\langle \hat{s}_1^z \hat{s}_2^z \hat{s}_3^z \rangle = b_1 b_2 b_3 + b_1' b_3 \delta_{12} + b_1' b_2 \delta_{13} + b_1 b_2' \delta_{23} + b_1'' \delta_{12} \delta_{13}, \tag{4.17}$$

where denoted

$$b_n = b(h_n), \qquad b_n' = \partial b(h_n) / \partial h_n, \qquad b_n'' = \partial^2 b(h_n) / \partial h_n^2. \tag{4.18}$$

Let us now consider the average values of arbitrary products $\hat{S}$ of spin operators $\hat{s}_n^+$, $\hat{s}_n^-$, $\hat{s}_n^z$ with the SO ρ_0 (4.10). First, we will prove that the average $\langle \hat{S} \rangle$ is different from zero only when $\hat{S}$ has the same number of operators $\hat{s}_n^+$, $\hat{s}_{n'}^-$ with arbitrary numbers of particles n, n'. For the proof, we proceed from the identity

$$e^{h\hat{s}^z} \hat{s}_n^{\pm} e^{-h\hat{s}^z} = \hat{s}_n^{\pm} e^{\pm h}, \tag{4.19}$$

which follows from (4.5) in the differential equation method. From here we have

$$e^{h\hat{s}^z} \hat{S} e^{-h\hat{s}^z} = \hat{S}\, e^{(N_+ - N_-)h}, \tag{4.20}$$

where N_+, N_- are quantities of $\hat{s}_n^+$ and $\hat{s}_{n'}^-$, respectively, in $\hat{S}$. Then we take into account relations

$$[\rho_0, \hat{b}^z] = 0, \tag{4.21}$$

(4.20), and the trace property (4.8), which gives

$$\mathrm{Sp}\,\rho_0 \hat{S} = \mathrm{Sp}\, e^{-h\hat{s}^z} \rho_0 e^{h\hat{s}^z} \hat{S} = \mathrm{Sp}\,\rho_0 e^{h\hat{s}^z} \hat{S} e^{-h\hat{s}^z} = e^{(N_+ - N_-)h}\, \mathrm{Sp}\,\rho_0 \hat{S}, \tag{4.22}$$

that is, if $\mathrm{Sp}\,\rho_0 \hat{S} \neq 0$, then $N_+ = N_-$.

Let us now consider a number of transformations that express the average of the product of spin operators through the average of the product of the one less number of operators.

$$\langle \hat{S}_1 \hat{s}_n^+ \hat{S}_2 \rangle = \langle [\hat{S}_1, \hat{s}_n^+] \hat{S}_2 \rangle + \mathrm{Sp}\,\rho_q s_n^+ \rho_q^{-1} \rho_q \hat{S}_1 \hat{S}_2 =$$
$$= \langle [\hat{S}_1, \hat{s}_n^+] \hat{S}_2 \rangle + \mathrm{Sp}\, e^{h_n \hat{s}_n^z} s_n^+ e^{-h_n \hat{s}_n^z} \rho_q \hat{S}_1 \hat{S}_2 = \langle [\hat{S}_1, \hat{s}_n^+] \hat{S}_2 \rangle + e^{h_n}\, \mathrm{Sp}\, s_n^+ \rho_q \hat{S}_1 \hat{S}_2 = \tag{4.23}$$

$$=\langle[\hat{S}_1,\hat{s}_n^+]\hat{S}_2\rangle+e^{h_n}\,\mathrm{Sp}\,\rho_q\hat{S}_1\hat{S}_2\hat{s}_n^+=\langle[\hat{S}_1,\hat{s}_n^+]\hat{S}_2\rangle+e^{h_n}\langle\hat{S}_1\hat{S}_2\hat{s}_n^+\rangle=$$

$$=\langle[\hat{S}_1,\hat{s}_n^+]\hat{S}_2\rangle+e^{h_n}\langle\hat{S}_1[\hat{S}_2,\hat{s}_n^+]\rangle+e^{h_n}\langle\hat{S}_1\hat{s}_n^+\hat{S}_2\rangle,$$

that is

$$\langle\hat{S}_1\hat{s}_n^+\hat{S}_2\rangle=\langle[\hat{S}_1,\hat{s}_n^+]\hat{S}_2\rangle+e^{h_n}\langle\hat{S}_1[\hat{S}_2,\hat{s}_n^+]\rangle+e^{h_n}\langle\hat{S}_1\hat{s}_n^+\hat{S}_2\rangle.$$

From here we finally have what was promised

$$\langle\hat{S}_1\hat{s}_n^+\hat{S}_2\rangle=-\mathrm{f}_n\langle[\hat{S}_1,\hat{s}_n^+]\hat{S}_2\rangle-(1+\mathrm{f}_n)\langle\hat{S}_1[\hat{S}_2,\hat{s}_n^+]\rangle$$
$$(\mathrm{f}_n\equiv\mathrm{f}(h_n),\qquad \mathrm{f}(h)\equiv(e^h-1)^{-1}).\tag{4.24}$$

This result is a generalization and simplification of the identity obtained earlier [16, 17]. Note that the function $\mathrm{f}(h)$ is expressed through the function $b(h)$, introduced in (4.15) for $s=1/2$, by the formula

$$\mathrm{f}=(1-2b)/4b\qquad(s=1/2)\tag{4.25}$$

since $e^{2x}=(1+\mathrm{th}x)/(1-\mathrm{th}x)$. Derivatives of the function b are also expressed through it by the formula

$$b'=1/4-b^2\qquad(s=1/2),\tag{4.26}$$

which is condemned by identity $(\mathrm{th}x)'=1/\mathrm{ch}^2x=1-\mathrm{th}^2x$.

Let's move on to the consideration of examples of the application of formula (4.24), which are important for the study of the Dicke superradiance model. For average products of two operators, we have

$$\langle\hat{s}_1^+\hat{s}_2^-\rangle=-(1+\mathrm{f}_1)\langle[\hat{s}_2^-,\hat{s}_1^+]\rangle=(1+\mathrm{f}_1)2\delta_{12}\langle\hat{s}_1^z\rangle=(1+\mathrm{f}_1)2\delta_{12}b_1,$$
$$\langle\hat{s}_1^-\hat{s}_2^+\rangle=-\mathrm{f}_2\langle[\hat{s}_1^-,\hat{s}_2^+]\rangle=\mathrm{f}_2 2\delta_{12}\langle\hat{s}_1^z\rangle=\mathrm{f}_1 2\delta_{12}b_1\tag{4.27}$$

Average products of three operators, taking into account (4.17) and (4.27), are

$$\langle\hat{s}_1^+\hat{s}_2^z\hat{s}_3^-\rangle=-(1+\mathrm{f}_1)\langle[\hat{s}_2^z\hat{s}_3^-,\hat{s}_1^+]\rangle=-(1+\mathrm{f}_1)\langle-\hat{s}_2^z2\delta_{13}\hat{s}_1^z\rangle-(1+\mathrm{f}_1)\langle\delta_{12}\hat{s}_1^+\hat{s}_2^-\rangle=$$

$$=2(1+\mathrm{f})\delta_{13}(b_1b_2+\delta_{12}b_1')-(1+\mathrm{f}_1)\delta_{12}\delta_{13}2(1+\mathrm{f}_3)b_3;$$

$$\langle\hat{s}_1^-\hat{s}_2^+\hat{s}_3^z\rangle=-\mathrm{f}_2\langle[\hat{s}_1^-,\hat{s}_2^+]\hat{s}_3^z\rangle-(1+\mathrm{f}_2)\langle\hat{s}_1^z[\hat{s}_3^z,\hat{s}_2^+]\rangle=\tag{4.28}$$

$$=-\mathrm{f}_2\langle(-2\delta_{12}\hat{s}_1^z)\hat{s}_3^z\rangle-(1+\mathrm{f}_2)\langle\hat{s}_1^-\hat{s}_2^+\delta_{23}\rangle=$$

$$=-2\mathrm{f}_2\delta_{12}(b_1b_3+\delta_{13}b_1')-(1+\mathrm{f}_2)\delta_{23}\delta_{12}2\mathrm{f}_1b_1;$$

$$\langle \hat{s}_1^+ \hat{s}_2^- \hat{s}_3^z \rangle = -(1+\mathrm{f}_1)\langle [\hat{s}_2^- \hat{s}_3^z, \hat{s}_1^+] \rangle = -(1+\mathrm{f}_1)\langle \hat{s}_2^- \delta_{31} \hat{s}_1^+ - 2 s_1^z \delta_{21} \hat{s}_3^z \rangle =$$

$$= -(1+\mathrm{f}_1)\delta_{31}\mathrm{f}_1 2\delta_{12} b_1 + 2(1+\mathrm{f}_1)\delta_{21}(b_1 b_3 + b_1' \delta_{13}) .$$

An average product of four operators equals

$$\langle \hat{s}_1^+ \hat{s}_2^- \hat{s}_3^+ \hat{s}_4^- \rangle = -(1+\mathrm{f}_1)\langle [\hat{s}_2^- \hat{s}_3^+ \hat{s}_4^-, \hat{s}_1^+] \rangle =$$

$$= -(1+\mathrm{f}_1)\langle [\hat{s}_2^-, \hat{s}_1^+] \hat{s}_3^+ \hat{s}_4^- + \hat{s}_2^- [\hat{s}_3^+, \hat{s}_1^+] \hat{s}_4^- + \hat{s}_2^- \hat{s}_3^+ [\hat{s}_4^-, \hat{s}_1^+] \rangle =$$

$$= -(1+\mathrm{f}_1)\langle -2\delta_{12} \hat{s}_2^z \hat{s}_3^+ \hat{s}_4^- - 2\delta_{14} \hat{s}_2^- \hat{s}_3^+ \hat{s}_4^z \rangle =$$

$$= -(1+\mathrm{f}_1)\langle 2\delta_{12}\mathrm{f}_3 [\hat{s}_2^z, \hat{s}_3^+] \hat{s}_4^- + 2\delta_{12}(1+\mathrm{f}_3)\hat{s}_2^z [\hat{s}_4^-, \hat{s}_3^+] \rangle -$$

$$-(1+\mathrm{f}_1)\langle 2\delta_{14}\mathrm{f}_3 [\hat{s}_2^-, \hat{s}_3^+] \hat{s}_4^z + 2\delta_{14}(1+\mathrm{f}_3)\hat{s}_2^- [\hat{s}_4^z, \hat{s}_3^+] \rangle = \tag{4.29}$$

$$= -(1+\mathrm{f}_1)\langle 2\delta_{12}\mathrm{f}_3 \delta_{23} \hat{s}_3^+ \hat{s}_4^- - 2\delta_{12}(1+\mathrm{f}_3)\hat{s}_2^z 2\delta_{34} s_3^z \rangle -$$

$$-(1+\mathrm{f}_1)\langle -2\delta_{14}\mathrm{f}_3 \delta_{23} 2\hat{s}_3^z \hat{s}_4^z + 2\delta_{14}(1+\mathrm{f}_3)\delta_{34} \hat{s}_2^- \hat{s}_3^+ \rangle =$$

$$= -2(1+\mathrm{f}_1)\mathrm{f}_3 \delta_{12}\delta_{23} \langle \hat{s}_3^+ \hat{s}_4^- \rangle + 4(1+\mathrm{f}_1)(1+\mathrm{f}_3)\delta_{12}\delta_{34}\langle \hat{s}_2^z s_3^z \rangle +$$

$$+ 4(1+\mathrm{f}_1)\mathrm{f}_3 \delta_{14}\delta_{23} \langle \hat{s}_3^z \hat{s}_4^z \rangle - 2(1+\mathrm{f}_1)(1+\mathrm{f}_3)\delta_{14}\delta_{34}\langle \hat{s}_2^- \hat{s}_3^+ \rangle .$$

Taking into account (4.17), (4.24), and (4.27), from here we have

$$\begin{gathered} \langle \hat{s}_1^+ \hat{s}_2^- \hat{s}_3^+ \hat{s}_4^- \rangle = \\ = -2(1+\mathrm{f}_1)\mathrm{f}_3 \delta_{12}\delta_{23}(1+\mathrm{f}_3)2\delta_{34} b_3 + 4(1+\mathrm{f}_1)(1+\mathrm{f}_3)\delta_{12}\delta_{34}(b_2 b_3 + \delta_{23} b_2') + \\ + 4(1+\mathrm{f}_1)\mathrm{f}_3 \delta_{14}\delta_{23}(b_3 b_4 + \delta_{34} b_3') - 2(1+\mathrm{f}_1)(1+\mathrm{f}_3)\delta_{14}\delta_{34}\mathrm{f}_2 2\delta_{23} b_2 , \end{gathered} \tag{4.30}$$

and

$$\begin{gathered} \langle (\hat{s}^+ \hat{s}^-)^2 \rangle = \langle \hat{s}^+ \hat{s}^- \hat{s}^+ \hat{s}^- \rangle = \sum_{1234} \langle \hat{s}_1^+ \hat{s}_2^- \hat{s}_3^+ \hat{s}_4^- \rangle = \\ = -4\sum_1 (1+\mathrm{f}_1)^2 \mathrm{f}_1 b_1 + 4\sum_{12}(1+\mathrm{f}_1)(1+\mathrm{f}_2) b_1 b_2 + 4\sum_1 (1+\mathrm{f}_1)^2 b_1' + \\ + 4\sum_{12}(1+\mathrm{f}_1)\mathrm{f}_2 b_1 b_2 + 4\sum_1 (1+\mathrm{f}_1)\mathrm{f}_1 b_1' - 4\sum_1 (1+\mathrm{f}_1)^2 \mathrm{f}_1 b_1 , \end{gathered}$$

that mean

$$\begin{gathered} \langle (\hat{s}^+ \hat{s}^-)^2 \rangle = \langle \hat{s}^+ \hat{s}^- \hat{s}^+ \hat{s}^- \rangle = \sum_{1234} \langle \hat{s}_1^+ \hat{s}_2^- \hat{s}_3^+ \hat{s}_4^- \rangle = \\ = -8\sum_1 (1+\mathrm{f}_1)^2 \mathrm{f}_1 b_1 + 4\sum_{12}(1+\mathrm{f}_1)(1+2\mathrm{f}_2) b_1 b_2 + 4\sum_1 (1+\mathrm{f}_1)(1+2\mathrm{f}_1) b_1' . \end{gathered} \tag{4.31}$$

Thus, rather complicated averages of products of spin operators can be calculated easily.

5. Dicke superradiance model dynamics

The Dicke model was proposed to describe the phenomenon of superradiance. In this model, the system consists of identical two-level atoms (emitters) and a system of photons (transverse electromagnetic field). The Dicke Hamiltonian of this system in terms of spin operators $\hat{\vec{s}}_n = (\hat{s}_n^x, \hat{s}_n^y, \hat{s}_n^z)$ and photon Bose operators c_k, c_k^+ has the form, according to (2.11),

$$\begin{gathered} \hat{H} = \hat{H}_0 + \hat{H}_1, \\ \hat{H}_0 = \sum_k \hbar\omega_k \hat{n}_k + \hbar\omega_0 \hat{s}^z, \qquad \hat{H}_1 = \sum_k \lambda_k (c_k + c_k^+)(\hat{s}^+ + \hat{s}^-), \\ \lambda_k \equiv \left(\frac{2\pi\hbar}{V}\right)^{1/2} \vec{e}_k \vec{e} d \qquad (\hat{n}_k \equiv a_k^+ a_k, \quad \hat{s}^z \equiv \sum_{1\le n\le N} \hat{s}_n^z, \quad \hat{s}^\pm \equiv \sum_{1\le n\le N} \hat{s}_n^\pm), \end{gathered} \tag{5.1}$$

where, to simplify notations and calculations, the states of photons are numbered with one index $k = (\alpha, \vec{k})$, the dipole moments of atoms are assumed to be the same, and therefore the coefficient λ_k does not depend on n.

The system state will be described, according to (3.1), with average values of operators

$$\hat{\eta}_k = \hat{n}_k, \qquad \hat{\eta}_1 = \hat{s}^z, \qquad \hat{\eta}_2 = \hat{s}^+ \hat{s}^-, \tag{5.2}$$

which are denoted by η_a: $\eta_k = n_k$, $\eta_1 = s_z$, η_2 and will be its RDPs.

All operators $\hat{\eta}_a$ from (5.2), as noted in (3.9), commute with themselves and with the main contribution to the Hamiltonian $\hat{H}_0$

$$[\hat{\eta}_a, \hat{\eta}_b] = 0, \qquad [\hat{H}_0, \hat{\eta}_a] = 0, \tag{5.3}$$

since, in particular,

$$[\hat{s}_n^z, \hat{s}_{n'}^\pm] = \pm \hat{s}_n^\pm \delta_{nn'}, \quad [\hat{s}_n^z, \hat{s}_{n'}^+ \hat{s}_{n''}^-] = \hat{s}_{n'}^+ \hat{s}_{n''}^- (\delta_{nn'} - \delta_{nn''}), \quad [\hat{\vec{s}}_n, \hat{n}_k] = 0, \quad [\hat{n}_k, \hat{n}_{k'}] = 0. \tag{5.4}$$

Relations (5.3) show that the Dicke model is a special case of the PYa model of the theory of non-equilibrium processes. The quasi-equilibrium SO ρ_q of the Dicke model taking into account all degrees of freedom can be written in the form

$$\rho_q = \rho_{qb} \rho_{qm}, \quad \rho_{qb} \equiv e^{\Omega_b - \sum_k Z_k \hat{n}_k}, \qquad \rho_{qm} \equiv e^{\Omega_m - Z_1 \hat{\eta}_1 - Z_2 \hat{\eta}_2}, \tag{5.5}$$

where ρ_{qb}, ρ_{qm} are the contributions of degrees of freedom of bosons (photons) and matter. Functions $\Omega_b(Z_{k'})$, $\Omega_m(Z_1, Z_2)$, $Z_k(n_{k'})$, $Z_1(\eta_1, \eta_2)$, $Z_2(\eta_1, \eta_2)$ in these expressions are determined by formulas

$$\begin{gathered} \mathrm{Sp}\,\rho_{qb} = 1, \quad \mathrm{Sp}\,\rho_{qm} = 1; \\ \mathrm{Sp}\,\rho_{qb} \hat{n}_k = n_k, \quad \mathrm{Sp}\,\rho_{qm} \hat{\eta}_1 = \eta_1, \quad \mathrm{Sp}\,\rho_{qm} \hat{\eta}_2 = \eta_2. \end{gathered} \tag{5.6}$$

Exact calculations of average values of the functions of spin operators with SO ρ_{qm} are impossible because the operator $\hat{\eta}_2$ in (5.2) is a quadratic form of spin operators (see [17] and Section 4).

However, calculating the averages is possible with a quasi-equilibrium statistical operator

$$\rho_q^0 = e^{\Omega^0 - Z_1^0 \hat{\eta}_1} = e^{\Omega^0 - Z_1^0 \sum_n \hat{s}_n^z}; \quad \mathrm{Sp}\rho_q^0 = 1, \quad \mathrm{Sp}\rho_q^0 \hat{\eta}_1 = \eta_1 \tag{5.7}$$

is possible (see [I3] and section 3) and the last two formulas give specific expressions for the functions

$$\Omega^0(Z_1^0), \qquad Z_1^0(\eta_1). \tag{5.8}$$

A way out for implementing the formalism of the PYa model in the theory of superradiance was proposed in our work [18], where instead of describing the state of the system with parameters η_1, η_2 we limited ourselves (along with its description at arbitrary values of η_1) to consider only the states with a parameter η_2 that differs from the average value in the state with the statistical operator ρ_q^0 by a small amount (deviation) $\delta\eta_2$

$$\delta\eta_2 \equiv \eta_2 - \eta_{20} = \mathrm{Sp}(\rho_{qm} - \rho_q^0)\hat{\eta}_2 \qquad (\eta_{20} \equiv \mathrm{Sp}\rho_q^0 \hat{\eta}_2; \quad \delta\eta_2 \sim \mu, \quad \mu \ll 1). \tag{5.9}$$

As a result, the quasi-equilibrium SO ρ_{qm} satisfies the equations

$$\mathrm{Sp}(\rho_{qm} - \rho_q^0)\hat{\eta}_1 = 0, \qquad \mathrm{Sp}(\rho_{qm} - \rho_q^0)\hat{\eta}_2 = \delta\eta_2, \qquad \mathrm{Sp}(\rho_{qm} - \rho_q^0) = 0, \tag{5.10}$$

which follow from the definitions (5.6), (5.7), and (5.9). When $\delta\eta_2$ is small, statistical operators ρ_{qm} and ρ_q^0, which are given by formulas (5.5) and (5.7), differ little from each other and values $\Omega - \Omega^0$, $Z_1 - Z_1^0$, and Z_2 are small. Therefore, from formulas (5.10) the quasi-equilibrium SO ρ_{qm} can be found in the form

$$\rho_{qm} = \rho_q^0 + \rho_q^0(F - A\hat{\eta}_1 - B\hat{\eta}_2) + O(\mu^2). \tag{5.11}$$

where F, A, B are values of the 1st order in μ. Substituting expression (5.11) into equation (5.10) gives a system of equations for the coefficients F, A, B

$$F x_1 - A x_{11} - B x_{12} = 0, \qquad F x_2 - A x_{12} - B x_{22} = \delta\hat{\eta}_2, \qquad F - A x_1 - B x_2 = 0 \tag{5.12}$$

where notations are introduced

$$x_a = \langle \hat{\eta}_a \rangle, \qquad x_{ab} = \langle \hat{\eta}_a \hat{\eta}_b \rangle; \qquad \langle \hat{S} \rangle \equiv \mathrm{Sp}\rho_q^0 \hat{S}. \tag{5.13}$$

Here and further, we leave the notation from Section 3 of the average value of an arbitrary spin operator by $\langle \hat{S} \rangle$, although ρ_q^0 is a special case of SO ρ_0 (see (4.9), (5.7)).

Equations (5.12) lead to such expressions for the values F, A, B

$$A = \frac{y_{12}}{y_{11}y_{22} - y_{12}^2}\delta\eta_2, \qquad B = \frac{y_{11}}{y_{12}^2 - y_{11}y_{22}}\delta\eta_2, \qquad F = \frac{x_2 y_{11} - x_1 y_{12}}{y_{12}^2 - y_{11}y_{22}}\delta\eta_2 \tag{5.14}$$

$$(y_{ab} \equiv \langle \hat{\eta}_a \hat{\eta}_b \rangle - \langle \hat{\eta}_a \rangle \langle \hat{\eta}_b \rangle).$$

Thus, formulas (5.5), (5.11), and (5.14) give the quasi-equilibrium SO ρ_q of the system in the form of expansion in powers of μ.

According to (3.4), (3.14), and (3.15) with taking into account (5.3), the time equations for RDPs η_a of the system state and its SO in the form of expansion in the powers of interaction $\hat{H}_1$ have the form

$$\partial_t \eta_a(t) = L_a(\eta(t)),$$
$$L_a(\eta) = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \operatorname{Sp}\rho_q[\hat{H}_1(\tau),[\hat{H}_1,\hat{\eta}_a]] + O(\lambda^3), \quad \hat{H}_1(\tau) \equiv e^{-i\tau\mathbf{L}_0}\hat{H}_1; \tag{5.15}$$
$$\rho(\eta) = \rho_q + \int_{-\infty}^{0} d\tau \frac{i}{\hbar}[\rho_q, \hat{H}_1(\tau)] + O(\lambda^2).$$

Here, it is taken into account that the first-order contribution to $L_a(\eta)$ is zero, since according to (5.3) and (5.5) $\operatorname{Sp}\rho_q[\hat{H}_1,\hat{\eta}_a] = 0$.

The time equations in terms of RDPs n_k, η_1, η_2

$$\partial_t n_k = L_k(n_{k'},\eta_1,\eta_2), \quad \partial_t \eta_1 = L_1(n_{k'},\eta_1,\eta_2), \quad \partial_t \eta_2 = L_2(n_{k'},\eta_1,\eta_2) \tag{5.16}$$

when considering small correlations in the system, respectively (5.9) have the form

$$\partial_t n_k = L_k(n_{k'},\eta_1,\eta_{20}+\delta\eta_2), \qquad \partial_t \eta_1 = L_1(n_{k'},\eta_1,\eta_{20}+\delta\eta_2),$$
$$\partial_t \delta\eta_2 = L_2(n_{k'},\eta_1,\eta_{20}+\delta\eta_2) - \frac{\partial \eta_{20}}{\partial \eta_1} L_1(n_{k'},\eta_1,\eta_{20}+\delta\eta_2). \tag{5.17}$$

Decomposing functions $L_a(n_{k'},\eta_1,\eta_{20}+\delta\eta_2)$ in a $\delta\eta_2$ power series

$$L_a(n_{k'},\eta_1,\eta_{20}+\delta\eta_2) = L_{a0}(n_{k'},\eta_1) + N_a(n_{k'},\eta_1)\delta\eta_2 + O(\mu^2), \tag{5.18}$$

we obtain the set of time equations of the theory in the linear approximation by $\delta\eta_2$

$$\partial_t n_k = L_{k0}(n_{k'},\eta_1) + N_k(n_{k'},\eta_1)\delta\eta_2 + O(\mu^2),$$
$$\partial_t \eta_1 = L_{10}(n_{k'},\eta_1) + N_1(n_{k'},\eta_1)\delta\eta_2 + O(\mu^2),$$
$$\partial_t \delta\eta_2 = [L_{20}(n_{k'},\eta_1) - \frac{\partial \eta_{20}}{\partial \eta_1} L_{10}(n_{k'},\eta_1)] + \tag{5.19}$$
$$+[N_2(n_{k'},\eta_1) - \frac{\partial \eta_{20}}{\partial \eta_1} N_1(n_{k'},\eta_1)]\delta\eta_2 + O(\mu^2).$$

The presence of the first term in the third equation requires research whether the value $\delta\eta_2$ will remain small over time.

6. Calculation of the elements of Dicke model dynamics

To calculate the right-hand sides of the equations for RDPs $L_a(\eta)$ according to (5.15), a quasiequilibrium SO is required, which is given by the formulas

$$\rho_q = \rho_{qb}\rho_{qm}, \qquad \rho_{qm} = \rho_q^0 + \rho_q^0(F - A\hat{\eta}_1 - B\hat{\eta}_2) + O(\mu^2). \tag{6.1}$$

The issue of calculating averages with a statistical operator ρ_q^0 was actually discussed in Section 3, where the more general SO ρ_0 was considered, for which $\rho_q^0 = \rho_0|_{h_n \to -Z_1^0}$ (see formulas (4.9), (5.7)). ρ_{qb} in (6.1) is the quasi-equilibrium SO of the photon system

$$\rho_{qb} = e^{\Omega_b - \sum_k Z_k \hat{n}_k};$$
$$\mathrm{Sp}\,\rho_{qb} = 1, \quad \mathrm{Sp}\,\rho_{qb}\hat{n}_k = n_k; \quad n_k = (e^{Z_k} - 1)^{-1}, \quad \Omega_b = \sum_k \ln(1 - e^{-Z_k}) \tag{6.2}$$

(see (5.5), (5.6)). Averages with SO ρ_{qb} are calculated using the Wick`s rules with elementary couplings

$$\overbrace{a_k^+ a_{k'}} = n_k \delta_{kk'}, \quad \overbrace{a_k a_{k'}^+} = (1+n_k)\delta_{kk'}, \quad \overbrace{a_k a_{k'}} = 0, \quad \overbrace{a_k^+ a_{k'}^+} = 0. \tag{6.3}$$

For further use of the quasi-equilibrium SO ρ_q (6.1) with accuracy up to and including contributions of the first order of μ, coefficients F, A, B should be calculated based on formulas (5.13) and (5.14). For our case $s = 1/2$, from (4.27), (4.28), and (4.31) we have

$$x_1 = \langle \hat{\eta}_1 \rangle = \sum_n \langle \hat{s}_n^z \rangle = bN,$$
$$x_{11} = \langle \hat{\eta}_1 \hat{\eta}_1 \rangle = \sum_{n_1, n_2} \langle \hat{s}_{n_1}^z \hat{s}_{n_2}^z \rangle = \sum_{n_1, n_2} (b^2 + b'\delta_{12}) = b^2 N^2 + b'N,$$
$$x_2 = \langle \hat{\eta}_2 \rangle = \sum_{n_1 n_2} \langle \hat{s}_{n_1}^+ \hat{s}_{n_2}^- \rangle = \sum_{n_1 n_2} (1+\mathrm{f})2b\delta_{12} = 2(1+\mathrm{f})bN,$$
$$x_{12} = \langle \hat{\eta}_1 \hat{\eta}_2 \rangle = \langle \hat{\eta}_2 \hat{\eta}_1 \rangle = \sum_{n_1 n_2 n} \langle \hat{s}_{n_1}^+ \hat{s}_{n_2}^- \hat{s}_{n_3}^z \rangle = \sum_{n_1 n_2 n_3} 2(1+\mathrm{f})\delta_{12}[b^2 + (b' - \mathrm{f}\,b)\delta_{13}] =$$
$$= 2(1+\mathrm{f})[b^2 N^2 + (b' - \mathrm{f}b)N] = 2(1+\mathrm{f})b^2 N^2 + (1+\mathrm{f})(b' - \mathrm{f}\,b)N,$$
$$x_{22} = \langle \hat{\eta}_2 \hat{\eta}_2 \rangle = \sum_{n_1 n_2 n_3 n_4} \langle \hat{s}_{n_1}^+ \hat{s}_{n_2}^- \hat{s}_{n_3}^+ \hat{s}_{n_4}^- \rangle = \tag{6.4}$$
$$= \sum_{n_1 n_2 n_3 n_4} [-4(1+\mathrm{f})^2 \mathrm{f}\,b\delta_{12}\delta_{23}\delta_{34} + 4(1+\mathrm{f})^2 \delta_{12}\delta_{34}(b^2 + \delta_{23}b') +$$
$$+ 4(1+\mathrm{f})\mathrm{f}\,\delta_{14}\delta_{23}(b^2 + \delta_{34}b') - 4(1+\mathrm{f})^2 \mathrm{f}\,b\delta_{14}\delta_{34}\delta_{23}] =$$
$$= \sum_{n_1 n_2 n_3 n_4} [4(1+\mathrm{f})^2 \delta_{12}\delta_{34}(b^2 + \delta_{23}b') + 4(1+\mathrm{f})\mathrm{f}\,\delta_{14}\delta_{23}(b^2 + \delta_{34}b') - 8(1+\mathrm{f})^2 \mathrm{f}\,b\delta_{14}\delta_{34}\delta_{23}] =$$
$$= 4(1+\mathrm{f})(1+2\mathrm{f})b^2 N^2 + 4(1+\mathrm{f})(1+2\mathrm{f})b'N - 8(1+\mathrm{f})^2 \mathrm{f}\,bN,$$

(here denoted $\delta_{12} \equiv \delta_{n_1 n_2} \ldots$), where according to (4.15), (4.18), and (4.24)

$$b = \frac{1}{2}\mathrm{th}\frac{h}{2}, \quad b' = \frac{\partial b(h)}{\partial h}, \quad \mathrm{f} \equiv (e^h - 1)^{-1} \tag{6.5}$$

with $h = -Z_1^0$. However, we do not actually need these expressions since the average $\langle \hat{\eta}_1 \rangle = \eta_1$ is a RDP in our description of the Dicke model dynamics. Therefore, formulas

$$b = \eta_1 / N\,, \qquad \mathrm{f} = (1-2b)/4b\,, \qquad b' = 1/4 - b^2\,, \tag{6.6}$$

are much more useful(see the first line in (6.4) and (4.25), (4.26)), which express all average values of an arbitrary spin operator $\langle \hat{S} \rangle$ through RDP η_1 and the number of atoms (emitters) N. Values necessary for calculating coefficients F, A, B based on formulas (5.13) and (5.14) according to (6.4) have the form

$$\begin{gathered} y_{11} = Nb'\,, \quad y_{12} = N(1+\mathrm{f})(b' - \mathrm{f}\,b)\,, \\ y_{22} = N4(1+\mathrm{f})[-2(1+\mathrm{f})\mathrm{f}\,b + (1+2\mathrm{f})b' + \mathrm{f}\,b^2] + N^2 4(1+\mathrm{f})\mathrm{f}\,b^2\,. \end{gathered} \tag{6.7}$$

Further analysis of coefficients F, A, B and their use in quasi-equilibrium SO will be discussed in the next paper. However, our description of nonequilibrium states of the Dicke model by parameters η_1, $\delta\eta_2$ (see (5.2) and (5.9)) can be considered implemented.

Let's proceed to the derivation of time equations for the RDPs of the Dicke model. The right-hand side $L_a(\eta)$ of the equations for RDPs has the form (5.15). Let's limit ourselves to contributions of the second order by interaction, additionally neglecting small correlations $\delta\eta_2$ in them. At the same time, the quasi-equilibrium SO ρ_q according to (6.1) is taken in the form $\rho_q = \rho_{qb}\rho_q^0$, that is, the expression will be calculated

$$L_{a0} \equiv -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \operatorname{Sp} \rho_{qb}\rho_q^0 [\hat{H}_1(\tau), [\hat{H}_1, \hat{\eta}_a]] \qquad (\hat{\eta}_a: \quad \hat{n}_k\,, \;\; \hat{\eta}_1 = \hat{s}^z\,, \;\; \hat{\eta}_2 = \hat{s}^+\hat{s}^-) \tag{6.8}$$

Note that the Hamiltonian of the Dicke model has the form (5.1).

Using the differential equation method, it is easy to prove formulas

$$\begin{gathered} e^{\frac{i}{\hbar}\tau\hat{H}_0} c_k e^{-\frac{i}{\hbar}\tau\hat{H}_0} = c_k e^{-i\omega_k\tau}\,, \qquad e^{\frac{i}{\hbar}\tau\hat{H}_0} c_k^+ e^{-\frac{i}{\hbar}\tau\hat{H}_0} = c_k^+ e^{i\omega_k\tau}\,, \\ e^{\frac{i}{\hbar}\tau\hat{H}_0} \hat{s}_n^+ e^{-\frac{i}{\hbar}\tau\hat{H}_0} = \hat{s}_n^+ e^{i\omega_0\tau}\,, \qquad e^{\frac{i}{\hbar}\tau\hat{H}_0} \hat{s}_n^- e^{-\frac{i}{\hbar}\tau\hat{H}_0} = \hat{s}_n^- e^{-i\omega_0\tau}\,, \end{gathered} \tag{6.9}$$

from which we find the expression

$$\hat{H}_1(\tau) = \sum_k \lambda_k (c_k e^{-i\omega_k\tau} + c_k^+ e^{i\omega_k\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau}) \tag{6.10}$$

that is necessary in (6.8). Simple calculations give also

$$\begin{gathered} [\hat{H}_1, n_k] = \lambda_k (c_k - c_k^+)(\hat{s}^+ + \hat{s}^-)\,, \qquad [\hat{H}_1, \hat{s}^z] = \sum_k \lambda_k (c_k + c_k^+)(-\hat{s}^+ + \hat{s}^-)\,. \\ [\hat{H}_1, \hat{s}^+\hat{s}^-] = 2\sum_k \lambda_k (c_k + c_k^+)(\hat{s}^+\hat{s}^z + \hat{s}^z\hat{s}^-)\,, \end{gathered} \tag{6.11}$$

which creates the basis for further calculations since

$$L_{n_k 0} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \operatorname{Sp} \rho_{qb}\rho_q^0 [\hat{H}_1(\tau), [\hat{H}_1, \hat{n}_k]]\,, \tag{6.12}$$

$$L_{10} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0[\hat{H}_1(\tau),[\hat{H}_1,\hat{s}^z]],$$

$$L_{20} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0[\hat{H}_1(\tau),[\hat{H}_1,\hat{s}^+\hat{s}^-]].$$

Very cumbersome calculations based on (6.8) can be simplified by a method, the essence of which is shown below. For the right-hand side of the equation $\partial_t n_k = L_{n_k}$ we use

$$L_{n_k 0} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k'} \lambda_{k'}[(c_{k'}e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau}),$$
$$\lambda_k(c_k - c_k^+)(\hat{s}^+ + \hat{s}^-)] =$$
$$= -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k'} \lambda_{k'}[(c_{k'}e^{-i\omega_{k}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau})_A(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})_B,$$
$$\kappa_k(c_k - c_k^+)_C(\hat{s}^+ + \hat{s}^-)_D] =$$
$$= -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k'} \lambda_k\lambda_{k'} \times$$
$$\{(c_k - c_k^+)_C(c_{k'}e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau})_A[(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})_B,(\hat{s}^+ + \hat{s}^-)_D] + \tag{6.13}$$
$$+[(c_{k'}e^{-i\omega_k\tau} + c_{k'}^+ e^{i\omega_k\tau})_A,(c_k - c_k^+)_C](\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})_B(\hat{s}^+ + \hat{s}^-)_D\} =$$
$$= -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k'} \lambda_k\lambda_{k'} \times$$
$$\{(c_k - c_k^+)(c_{k'}e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau})[(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau}),(\hat{s}^+ + \hat{s}^-)] +$$
$$+[(c_{k'}e^{-i\omega_k\tau} + c_{k'}^+ e^{i\omega_k\tau}),(c_k - c_k^+)](\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})(\hat{s}^+ + \hat{s}^-)\},$$

where the relation

$$[AB,CD] = CA[B,D] + C[A,D]B + A[B,C]D + [A,C]BD \tag{6.14}$$

and commutativity of photon operators and spin ones are taken into account.

Similarly for the right-hand side L_{10} of the equation $\partial_t \eta_1 = L_1$

$$L_{10} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k,k'} \lambda_k\lambda_{k'} \times$$
$$\{(c_k + c_k^+)_C(c_{k'}e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau})_A[(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})_B,(-\hat{s}^+ + \hat{s}^-)_D] + \tag{6.15}$$
$$+[(c_{k'}e^{-i\omega_k\tau} + c_{k'}^+ e^{i\omega_k\tau})_A,(c_k + c_k^+)_C](\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})_B(-\hat{s}^+ + \hat{s}^-)_D\}$$

and for that of the equation $\partial_t \eta_2 = L_2$

$$L_{20} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k,k'} \lambda_k \lambda_{k'} \times$$
$$\times 2\{(c_k - c_k^+)_C (c_{k'} e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau})_A [(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})_B, (\hat{s}^+\hat{s}^z + \hat{s}^z\hat{s}^-)_D] + \qquad (6.16)$$
$$+[(c_{k'} e^{-i\omega_k\tau} + c_{k'}^+ e^{i\omega_k\tau})_A, (c_k - c_k^+)_C](\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})_B (\hat{s}^+\hat{s}^z + \hat{s}^z\hat{s}^-)_D\}$$

Calculating the commutators in expressions for L_{a0} above

$$L_{n_k 0} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k'} \lambda_k \lambda_{k'} \times$$
$$\{(c_k - c_k^+)(c_{k'} e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau}) 2\hat{s}^z (e^{i\omega_0\tau} - e^{-i\omega_0\tau}) +$$
$$-\delta_{kk'}(e^{-i\omega_k\tau} + e^{i\omega_k\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})(\hat{s}^+ + \hat{s}^-)\},$$

$$L_{10} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k,k'} \lambda_k \lambda_{k'} \times$$
$$\{(c_k + c_k^+)(c_{k'} e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau}) 2\hat{s}^z (e^{i\omega_0\tau} + e^{-i\omega_0\tau}) + \qquad (6.17)$$
$$+\delta_{kk'}(e^{-i\omega_k\tau} - e^{i\omega_k\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})(-\hat{s}^+ + \hat{s}^-)\},$$

$$L_{20} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_{qb}\rho_q^0 \sum_{k,k'} \lambda_k \lambda_{k'} \times$$
$$\times 2\{(c_k - c_k^+)(c_{k'} e^{-i\omega_{k'}\tau} + c_{k'}^+ e^{i\omega_{k'}\tau})\{(-\hat{s}^+\hat{s}^+ - \hat{s}^+\hat{s}^- + 2\hat{s}^z\hat{s}^z) e^{i\omega_0\tau} +$$
$$+(\hat{s}^-\hat{s}^- + \hat{s}^-\hat{s}^+ - 2\hat{s}^z\hat{s}^z) e^{-i\omega_0\tau}\} -$$
$$-\delta_{kk'}(e^{-i\omega_k\tau} + e^{i\omega_k\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})(\hat{s}^+\hat{s}^z + \hat{s}^z\hat{s}^-)\}$$

and then photon averages according to Wick's rules, we obtain

$$L_{n_k 0} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_q^0 \lambda_k^2 \times$$
$$\times\{\{-e^{-i\omega_k\tau} n_k + e^{i\omega_k\tau}(1+n_k)\} 2\hat{s}^z (e^{i\omega_0\tau} - e^{-i\omega_0\tau}) +$$
$$-(e^{-i\omega_k\tau} + e^{i\omega_k\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})(\hat{s}^+ + \hat{s}^-)\},$$
$$(6.18)$$
$$L_{10} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \mathrm{Sp}\,\rho_q^0 \sum_k \lambda_k^2 \times$$
$$\times\{\{n_k e^{-i\omega_k\tau} + (1+n_k) e^{i\omega_k\tau}\} 2\hat{s}^z (e^{i\omega_0\tau} + e^{-i\omega_0\tau}) +$$
$$+(e^{-i\omega_k\tau} - e^{i\omega_k\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})(-\hat{s}^+ + \hat{s}^-)\},$$

$$L_{20} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \,\mathrm{Sp}\,\rho_q^0 \sum_k \lambda_k^2 \times$$
$$\times 2\{\{-n_k e^{-i\omega_k\tau} + (1+n_k)e^{i\omega_k\tau}\}\{(-\hat{s}^+\hat{s}^+ - \hat{s}^+\hat{s}^- + 2\hat{s}^z\hat{s}^z)e^{i\omega_0\tau} +$$
$$+(\hat{s}^-\hat{s}^- + \hat{s}^-\hat{s}^+ - 2\hat{s}^z\hat{s}^z)e^{-i\omega_0\tau}\} -$$
$$-(e^{-i\omega_k\tau} + e^{i\omega_k\tau})(\hat{s}^+ e^{i\omega_0\tau} + \hat{s}^- e^{-i\omega_0\tau})(\hat{s}^+\hat{s}^z + \hat{s}^z\hat{s}^-)\}\,.$$

Final calculations of spin averages with a quasi-equilibrium distribution ρ_q^0 in the right-hand sides of time equations for RDPs give

$$L_{n_k 0} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \lambda_k^2 \times$$
$$\{\{-e^{-i\omega_k\tau} n_k + e^{i\omega_k\tau}(1+n_k)\}2\langle\hat{s}^z\rangle(e^{i\omega_0\tau} - e^{-i\omega_0\tau}) +$$
$$-(e^{-i\omega_k\tau} + e^{i\omega_k\tau})(\langle\hat{s}^+\hat{s}^-\rangle e^{i\omega_0\tau} + \langle\hat{s}^-\hat{s}^+\rangle e^{-i\omega_0\tau})\}\,,$$

$$L_{10} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \sum_k \lambda_k^2 \times$$
$$\{\{n_k e^{-i\omega_k\tau} + (1+n_k)e^{i\omega_k\tau}\}2\langle\hat{s}^z\rangle(e^{i\omega_0\tau} + e^{-i\omega_0\tau}) +$$
$$+(e^{-i\omega_k\tau} - e^{i\omega_k\tau})(\langle\hat{s}^+\hat{s}^-\rangle e^{i\omega_0\tau} - \langle\hat{s}^+\hat{s}^-\rangle e^{-i\omega_0\tau})\}\,, \tag{6.19}$$

$$L_{20} = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau \sum_k \lambda_k^2 \times$$
$$\times 2\{\{-n_k e^{-i\omega_k\tau} + (1+n_k)e^{i\omega_k\tau}\}\{(-\langle\hat{s}^+\hat{s}^-\rangle + 2\langle\hat{s}^z\hat{s}^z\rangle)e^{i\omega_0\tau} +$$
$$+(\langle\hat{s}^-\hat{s}^+\rangle - 2\langle\hat{s}^z\hat{s}^z\rangle)e^{-i\omega_0\tau}\} -$$
$$-(e^{-i\omega_k\tau} + e^{i\omega_k\tau})(\langle\hat{s}^+\hat{s}^z\hat{s}^-\rangle e^{i\omega_0\tau} + \langle\hat{s}^-\hat{s}^+\hat{s}^z\rangle e^{-i\omega_0\tau})\}\,,$$

where relations are taken into account

$$\langle\hat{s}^+\hat{s}^-\rangle = 2(1+\mathrm{f})bN\,, \qquad \langle\hat{s}^-\hat{s}^+\rangle = 2\mathrm{f}bN\,,$$
$$\langle\hat{s}^+\hat{s}^z\hat{s}^-\rangle = 2(1+\mathrm{f})b^2N^2 + 2(1+\mathrm{f})b'N + 2(1+\mathrm{f})^2 bN\,, \tag{6.20}$$
$$\langle\hat{s}^-\hat{s}^+\hat{s}^z\rangle = -2\mathrm{f}b^2N^2 - 2\mathrm{f}b'N - (1+\mathrm{f})\mathrm{f}b^2N\,,$$

which follow from (4.27) and (4.28). Formulas (6.6), (6.19), and (6.20) display that the right-hand sides L_{a0} of time equations for RDPs are expressed through the degree of excitation of atoms η_1 and their quantity N.

In the previous formulas, we used the periodic boundary conditions. В попередніх формулах нами використані періодичні крайові умови. Therefore, in (6.19) in the sums over $\vec{k}$ it is necessary to perform a limit transition with taking into account the expression (2.11) for $\lambda_{\vec{k}\alpha}$

$$\sum_{\vec{k}\alpha}\lambda^2_{\alpha\vec{k}}\varphi(n_{\alpha\vec{k}})=\frac{2\pi\hbar}{V}\sum_{\vec{k}\alpha}(\vec{e}_{\alpha\vec{k}}\vec{e})^2\varphi(n_{\alpha\vec{k}})\xrightarrow[V\to\infty]{}\frac{\hbar}{(2\pi)^2}\sum_{\alpha}\int d^3k(\vec{e}_{\alpha\vec{k}}\vec{e})^2\varphi(n_{\alpha\vec{k}}) \tag{6.21}$$

(V is a field volume, $\varphi(n_{\alpha\vec{k}})$ is some function) considering that the electromagnetic field covers the entire space.

The integrals over τ in (6.19) are also taken because

$$\mathrm{Re}\int_{-\infty}^{0}d\tau e^{i\omega\tau}=\mathrm{Re}\lim_{\eta\to+0}\int_{-\infty}^{0}d\tau e^{(\eta+i\omega)\tau}=\mathrm{Re}\lim_{\eta\to+0}\frac{\eta-i\omega}{\eta^2+\omega^2}=\pi\delta(\omega)\,. \tag{6.22}$$

That is why, finally, the right-hand sides of the equations for RDPs take the form

$$L_{n_{\alpha k}0}=\frac{2\pi^2}{\hbar}\frac{1}{V}(\vec{e}_{\alpha\vec{k}}\vec{e})^2\,\delta(\omega_k-\omega_0)\{(1+2n_{\alpha\vec{k}})2\langle\hat{s}^z\rangle+\langle\hat{s}^+\hat{s}^-\rangle+\langle\hat{s}^-\hat{s}^+\rangle\}\,,$$

$$\begin{aligned}
L_{10}&=-\frac{\hbar}{4\pi}\sum_{\alpha}\int d^3k(\vec{e}_{\alpha\vec{k}}\vec{e})^2\,\delta(\omega_k-\omega_0)\{(1+2n_{\alpha\vec{k}})2\langle\hat{s}^z\rangle+\langle\hat{s}^+\hat{s}^-\rangle+\langle\hat{s}^+\hat{s}^-\rangle\}\,,\\
L_{20}&=\frac{\hbar}{2\pi}\sum_{\alpha}\int d^3k(\vec{e}_{\alpha\vec{k}}\vec{e})^2\,\delta(\omega_k-\omega_0)\{n_{\alpha\vec{k}}(\langle\hat{s}^+\hat{s}^-\rangle+\langle\hat{s}^-\hat{s}^+\rangle-4\langle\hat{s}^z\hat{s}^z\rangle)+\\
&\qquad+\langle\hat{s}^+\hat{s}^z\hat{s}^-\rangle+\langle\hat{s}^-\hat{s}^+\hat{s}^z\rangle\}\,,
\end{aligned} \tag{6.23}$$

where the spin averages are given in (6.20). Functions (6.23) in accordance with (6.19) determine the dynamics of the RDPs $n_{\alpha\vec{k}},\eta_1$ without taking into account small correlations $\delta\eta_2$ between atoms, and also allow to investigate under which conditions the parameter $\delta\eta_2$ remains small over time. Consideration these issues and the role of correlations in the dynamics of the Dicke model will be the subject of the next work.

7. Conclusions

In the paper the construction of the Hamiltonian for the quantum system of atoms and the electromagnetic field is analyzed for the case of atom system, dimensions of which are much smaller than the characteristic wavelength of atom emission. In such approximation the system Hamiltonian does not depend on atom coordinates. Atoms are considered in the two-level approximation and interact with field via dipole-electric mechanism, and this leads to the Dicke model [1]. In it, the notion “electric dipole moment of atom” is used despite its average value in the atom eigenstates are zero. The correct formulation is that the interaction contains the dipole moment operator in the two-level atom space. Our study shows that the system of atoms in the Dicke model behaves as a system of fixed in the space particles with spin $s=1/2$, which interact with photons.

The leading idea of our work is applying Bogolyubov reduced description method to non-equilibrium states in Dicke model dynamics with using Peletminsky–Yatsenko method scheme. The course of the Dicke superradiant process is highly dependent of the initial state of the system. In the specified method this aspect is analyzed with effective initial conditions for the Cauchy problem. In this paper, the detailed derivation of the integral equation for such conditions is presented.

The Peletminsky–Yatsenko approach operates with a quasi-equilibrium statistical operator ρ_q depending on reduced description parameters, and the simplest operator describing correlations in atom subsystem $\hat{\eta}_2$, which is the quadratic form of spin operators, makes it impossible to calculate necessary averages. In this paper, the development of our previous ideas of overcoming the corresponding difficulties through the consideration of the small deviation $\delta\eta_2 = \eta_2 - \eta_{20}$ of the RDP η_2 from its value at using only the linear form of spin operators. In the present paper the calculation of ρ_q in the perturbation theory by $\delta\eta_2$ is simplified. This theory implemented in the technique of calculations with spin operators.

We gave the underlined attention to applying the achievements of the theory of spin (magnetic) systems. Our work includes some improvements of the results of [16] concerning the calculation of averages for operators, which are products of operators $\hat{s}_n^z, \hat{s}_n^{\pm}$ (n is an atomic number, $1 \le n \le N$, N denotes the quantity of atoms). Averages were calculated in the state described as a spin system in an external field that proved to be equivalent to the system of atoms with fixed excitations h_n. We obtained a simple formula reducing the averaged value of the product of a certain number of operators to the products of one less operator quantity. As a result, all averages can be expressed via averaged products of $\hat{s}_n^z$ operators. Herewith the averages $\langle \hat{s}_{n_1}^z ... \hat{s}_{n_m}^z \rangle$ are expressed through the function $b(h)$ and its derivatives and $\langle \hat{s}_n^z \rangle = b(h_n)$ according to the definition.

For our study of the Dicke model, the spin $s = 1/2$ case is important. It is proved that in this case the necessary function from the reduction formula is expressed through the function $b(h)$ and the identity $b' = 1/4 - b^2$ is true. The aforementioned developments in the theory of spin systems actually make it possible to implement the theory of non-equilibrium states of the Dicke model of superradiance in the framework of the RDM in the final sections of our paper.

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