DICKE MODEL INVESTIGATIONS IN THE FRAMEWORK OF REDUCED DESCRIPTION METHOD

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The reduced description method (RDM) is based on Bogolyubov's idea that at large time the non-equilibrium state evolution of a statistical system can be described with the limited number of parameters. The way to the right choice of such parameters and constructing the equations of time evolution for them was opened by the works of Kharkiv school in statistical physics [1]. Since early 2000-ies the authors deal with applying the proposed technique to Dicke superradiance – the unique phenomenon of emitter system self-organization in the process of reaching the equilibrium state from excited one [2, 3]. We are interested in a more detailed picture of correlation development both in emitter and field subsystems. The problem of correlator decoupling which arises in the Bogolyubov method of boson variable elimination seems to need attention. In RDM, including the binary correlation functions into the set of reduced description parameters results in the necessity of calculating the averages with quasi-equilibrium Hamiltonians where such new parameters are present. Usually, two-level electromagnetic emitters are described using the quasispin operators constructed with Pauli matrices. While considering the acoustic superradiance, spin and phonon operators are necessary for the Hamiltonian construction. The operator forms prove to be the same for boson fields of different nature. Thus, we face the problem of averaging in the case when the exponential statistical operator includes a quadratic form of spin operator in the exponent index.

Keywords: emission, quasispin, Hamiltonian, superradiance, reduced description parameter.

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1. Introduction

Since the early 2000-ies a few scientists of the Quantum Macrophysics Department started investigating the Dicke model of a quantum system of two-level emitters interacting via electromagnetic field. In the pioneer paper by Dicke [1], it has been shown proceeding from the analysis of the quantum state evolution that such a system reveals an unusual way of relaxation from the excited state to the ground one. In this process, the self-organization in the emitter system takes place and the radiation energy is emitted in the form of a short coherent pulse. Unlike the laser mechanism of the coherent generation, when a stimulated emission in the resonating cavity provides the necessary effect, the phenomenon predicted by Dicke is based on the spontaneous emission and does not need any resonator. It paves the way to the coherent electromagnetic wave generation in the frequency range where mirrors are absent, namely X-rays and even γ-radiation [2]. In the case of Dicke superradiance, opposed to such quantum phenomena as superfluidity and superconductivity, the theory prediction was ahead of experimental observations, the first of which dates to 1973 [3]. The prospects of military application aroused great interest tosuperradiance study in 1980-ies, but the general scientific value of this phenomenon as the most non-trivial example of relaxation process is the main basis for the development of research work concerning superradiant systems. Several advanced theoretical methods were created in connection with the problem of coherent spontaneous electromagnetic emission. Both equilibrium and nonequilibrium properties of the Dicke model were studied. The analogous phenomena in acoustics were under consideration as tools for obtaining some information about crystals and controlling their states [4].

The dynamics of Dicke model was investigated with different approaches: collective operators (the case of a concentrated system, i.e. with linear dimensions much less than the wavelength of generated emission) [1], correlation development consideration (applicable for a prolonged system) [5], Markoff kinetic equations with projection operator method [6, 7]. Bogolyubov method of boson variable elimination [8, 9] proved to be very effective in the superradiance theory development. In this method, the electromagnetic field plays the role of a bosonic thermostat. Thus, another method (also connected with N. N. Bogolyubov's ideas) seemed to be an attractive tool for studying the Dicke process. We mean the reduced description method in the formulation of Kharkiv scientists S. Peletminskii and A. Yatsenko where the variable circle can be expanded. In this method, the kinetic equations for a nonequilibrium system are constructed from the condition that at big times the system state is described with a restricted quantity of variables. The program of such investigation was put into life and this paper presents the review of the obtained results.

2. Hamiltonian structure of the Dicke model

The most general structure of the Dicke Hamiltonian is

$$\hat{H} = \hat{H}_{\mathrm{m}} + \hat{H}_{\mathrm{f}} + \hat{H}_{\mathrm{mf}} \qquad (1)$$

where index m denotes the emitter subsystem – matter, f – boson field, mf relates to their interaction. The Hamiltonian of two-level emitters can be written with Pauli matrices σ_z multiplied by $\hbar\Omega/2$ (Ω is an operation transition frequency). If electromagnetic field is under consideration, the operator of the boson field has the form

$$\hat{H}_{\mathrm{f}} = \sum_{\alpha,\mathbf{k}} \hbar\omega_k c^{+}_{\alpha\mathbf{k}} c_{\alpha\mathbf{k}} \qquad (\omega_k = ck, \quad k \equiv |\mathbf{k}|) \qquad (2)$$

including usual bosonic operators numerated with photon momentum $\mathbf{k}$ and polarization α. The problem of correct matter-field interaction description requires special attention. Constructing the interaction Hamiltonian proceeding from the generalized momentum $\mathbf{p} - \frac{e}{c}\mathbf{A}$ for a non-relativistic particle with a charge e in the field described with vector potential $\mathbf{A}$ faces the question concerning the gauge invariance of results and omitting the term proportional to $\mathbf{A}^2$ [9]. These problems were discussed in detail in the monograph [10] with the affirmation about some contradictions between the results obtained with dipole approximation $\mathbf{x}\cdot\mathbf{E}$ Hamiltonian and with $\mathbf{p}\cdot\mathbf{A}$ Hamiltonian using the Coulomb gauge of electromagnetic field. Such difficulties were overcome in our paper [11] based on the fundamental Lagrangian formulation of classical electrodynamics and further quantization procedure. We considered the system of one-electron atoms (numerated by index a) in an external electromagnetic field $(\varphi, \mathbf{A})$. Its Lagrange function after an

obvious expansion in a series in electron radius-vectors $\mathbf{x}_{ae}$ relative to the atomic nucleus takes the form

$$L(t)=\sum_a\left(\frac{M\mathbf{u}_a^2}{2}+\frac{m\mathbf{u}_{ae}^2}{2}+\frac{e^2}{|\mathbf{x}_{ae}|}\right)+\sum_a e\mathbf{x}_{ae}\cdot\frac{\partial\varphi(\mathbf{x}_a,t)}{\partial\mathbf{x}_a}- \\ -\frac{e}{c}\sum_a\left(\mathbf{u}_{ae}\cdot\mathbf{A}(\mathbf{x}_a,t)+\mathbf{u}_{an}\mathbf{x}_{ael}\frac{\partial\mathbf{A}(\mathbf{x}_a,t)}{\partial\mathbf{x}_{al}}\right) \tag{3}$$

where M and m are atom nucleus and electron masses, $\mathbf{u}_a$ and $\mathbf{u}_{ae}$ are a-th atom nucleus and electron (relative) velocities, correspondingly, n and l are vector indices. Notice that in (3) electron charge equals $-e$, hence a nucleus charge is e. This expression leads to the Hamilton function, which, when using generalized momenta $\mathbf{p}_a=M\mathbf{u}_a-\frac{e}{c}\mathbf{x}_{ael}\frac{\partial\mathbf{A}_n(\mathbf{x}_a,t)}{\partial\mathbf{x}_{al}}$ and $\mathbf{p}_{ae}=m\mathbf{u}_{ae}-\frac{e}{c}\mathbf{A}(\mathbf{x}_a,t)$ as well as atom dipole momentum $\mathbf{d}_a=-e\mathbf{x}_{ae}$, has the form

$$H_1=\sum_a\left(\frac{\mathbf{p}_{ae}^2}{2m}-\frac{e^2}{|\mathbf{x}_{ae}|}\right)+\sum_a\frac{\mathbf{p}_a^2}{2M}-\sum_a\mathbf{d}_a\cdot\mathbf{E}(\mathbf{x}_a,t) \tag{4}$$

if vector potential $\mathbf{A}$ is considered equal to zero. Though we obtain the dipole approximation, its substantiation is invalid in the usual weak relativistic assumption $\mathbf{A}(\mathbf{x},t)=0,\ \varphi(\mathbf{x},t)\neq 0$. The possible application of the Coulomb gauge $\varphi(\mathbf{x},t)=0$, $\mathrm{div}\mathbf{A}(\mathbf{x},t)=0$ results in the Hamilton function

$$H_2=\sum_a\left(\frac{1}{2m}\left(\mathbf{p}_{ae}+\frac{e}{c}\mathbf{A}(\mathbf{x}_a,t)\right)^2-\frac{e^2}{|\mathbf{x}_{ae}|}\right)+\sum_a\frac{1}{2M}\left(\mathbf{p}_a-\frac{1}{c}\mathbf{d}_{al}\frac{\partial\mathbf{A}(\mathbf{x}_a,t)}{\partial\mathbf{x}_{al}}\right)^2 \tag{5}$$

containing vector-potential derivatives.

Substituting operators for classical dynamical variables, we come to the Hamilton operators $\hat{H}_1$ and $\hat{H}_2$ for quantum particles in a classical field. The evolution of systems with such Hamiltonians is described by Liouville equations

$$\frac{\partial\rho_n}{\partial t}=-\frac{i}{\hbar}\left[\hat{H}_n,\rho_n\right].\qquad n=1,2 \tag{6}$$

We have shown that statistical operators ρ_1 i ρ_2 are related by unitary transformation

$$\rho_1(t)=\hat{U}(t)\rho_2(t)\hat{U}^{\dagger}(t),\qquad \hat{U}(t)\equiv\exp\left(-i\frac{e}{\hbar c}\sum_a\hat{\mathbf{x}}_{ae}\cdot\mathbf{A}(\mathbf{x}_a,t)\right). \tag{7}$$

This fact confirms the equivalence of the Hamiltonians of two types and is ensured by equality

$$\hat{H}_1(t)=\hat{U}(t)\hat{H}_2(t)\hat{U}^{\dagger}(t)+\sum_a \mathbf{d}_a\cdot\frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x}_a,t)}{\partial t}. \qquad (8)$$

Since in the Coulomb gauge $\mathbf{E}(\mathbf{x}_a,t)=-\frac{1}{c}\frac{\partial\mathbf{A}(\mathbf{x}_a,t)}{\partial t}$, we can use the more convenient Hamiltonian form (4) in the quantum picture. The right analysis based on the Hamiltonian of $\mathbf{p}\cdot\mathbf{A}$ type (5) requires using the statistical operator ρ_2 and transformed physical variable operators $\hat{U}^{\dagger}(t)O(t)\hat{U}(t)$ [11].

The next problem is the interaction term H_{mf} view in the Dicke Hamiltonian. Restricting ourselves with two-level objects, we must find the matrix elements of the operator

$$\hat{H}_{ae}=-\hat{\mathbf{d}}_a\cdot\mathbf{E}(\mathbf{x}_a,t), \qquad (9)$$

for which the matrix elements of the operator $\hat{\mathbf{d}}_a$ in the space of eigenstates of the operator $\hat{H}_{\mathrm{m}a}$, i.e. $|a,-\rangle, |a,+\rangle$, are necessary. The results for an arbitrary emitter are

$$\langle a,-|\hat{\mathbf{d}}_a|a,-\rangle=0,\quad \langle a,+|\hat{\mathbf{d}}_a|a,+\rangle=0,\quad \langle a,+|\hat{\mathbf{d}}_a|a,-\rangle=\mathbf{d}_a,\quad \langle a,-|\hat{\mathbf{d}}_a|a,+\rangle=\mathbf{d}_a^{*}. \qquad (10)$$

The diagonal matrix elements are zero because $\hat{\mathbf{d}}_a$ is an odd operator and eigenstates have certain parity. Non-diagonal matrix elements are complex conjugated vectors with the same modulus for all atoms and an arbitrary phase because of this value arbitrariness in each atom. Their space orientation is also arbitrary if there are no reasons for a selected direction of the quantization axis. In literature, the proposed form of $\hat{\mathbf{d}}_a$ is $\mathbf{d}_r\sigma_x-\mathbf{d}_i\sigma_y$ where $\mathbf{d}_r, \mathbf{d}_i$ are certain real vectors to be concretized in each case [12]. We propose a more perfect form:

$$\hat{\mathbf{d}}_a=d\,\sigma_{ax}\mathbf{n}_a e^{i\varphi_a} \qquad (11)$$

where $\mathbf{n}_a$ are randomly oriented unitary vectors. The rotational invariance of their distribution will be taken into account further. The phase of dipole moment oscillation is an important parameter in the classical picture of superradiance, but in the quantum consideration the only vector $\mathbf{n}_a$ seems to be enough.

Omitting kinetic energy of emitters, we write the final view of Dicke Hamiltonian as

$$\hat{H}_{\mathrm{D}}=\sum_a\left(\hbar\Omega\hat{R}_{az}-2\hat{R}_{ax}\left(\hat{\mathbf{d}}_a\cdot\hat{\mathbf{E}}^{t}(\mathbf{x}_a)\right)\right)+\sum_{\alpha,\mathbf{k}}\hbar\omega_k c^{+}_{\alpha\mathbf{k}}c_{\alpha\mathbf{k}} \qquad (12)$$

using Dicke quasispin operators $\hat{R}_{al}=\frac{1}{2}\sigma_{al}$, $(l=x,y,z)$ [1] and transversal electric field operator

$$\hat{\mathbf{E}}^t(\mathbf{x}) = \sum_{\mathbf{k},\alpha}\left(\frac{2\pi\hbar\omega_k}{V}\right)^{1/2}\left(\mathbf{e}_\alpha(\mathbf{k})c_{\alpha\mathbf{k}}e^{i\mathbf{k}\cdot\mathbf{x}} + \mathbf{e}^*_\alpha(\mathbf{k})c^+_{\alpha\mathbf{k}}e^{-i\mathbf{k}\cdot\mathbf{x}}\right) \quad (13)$$

where the complex polarization vectors $\mathbf{e}_\alpha(\mathbf{k})$ correspond to the circular field polarization.

3. Reduced description parameters for the Dicke model

The reduced description method (RDM) is based on the idea that at large time the statistical operator (SO) of a nonequilibrium system $\rho(t)$ can be regarded as a function of the restricted number of observables – reduced description parameters (RDPs) η: $\rho(t)\xrightarrow[t>>\tau_0]{}\rho(\eta(t,\rho_0))$, and then the relation $\mathrm{Sp}\rho(t)\hat{\eta}_a \xrightarrow[t>>\tau_0]{} \eta_a(t,\rho_0)$ is valid (here and in equations (14) – (18) *a* numerates RDPs). In this case, we can describe the system evolution with differential equations for RDPs

$$\partial_t\eta_a(t,\rho_0) = L_a(\eta(t,\rho_0)), \qquad L_a(\eta) \equiv \mathrm{Sp}\,\hat{\eta}_a\hat{L}\rho(\eta) \quad (14)$$

proceeding from the Liouville equation

$$\partial_t\rho(\eta(t,\rho_0)) = \frac{i}{\hbar}\left[\hat{H},\rho\right] \equiv \hat{L}\rho(\eta(t,\rho_0)) \quad (15)$$

where $\hat{H}$ is a system Hamiltonian. Sincethe transition to the reduced description takes place at large time, the right initial values for RDPs should be chosen (see [13]).

Constructing the real evolution picture in this framework is possible if we deal with a system including well-studied subsystems with relatively weak interaction between them ($\hat{H} = \hat{H}_0 + \hat{V}$). The limiting form of the RDM Hamiltonian is a quasi-equilibrium one $\rho^{(0)}(\eta) = \exp\left\{\Omega(\eta) - \sum_a Y_a(\eta)\hat{\eta}_a\right\}$ ensuring the equalities $\mathrm{Sp}\hat{\eta}_a\rho^{(0)}(\eta) = \eta_a$ and $\mathrm{Sp}\rho^{(0)}(\eta) = 1$. Such requirements result in the integral equation

$$\rho(\eta) = \rho^{(0)}(\eta) - i\int_{-\infty}^{0} d\tau e^{i\hat{H}_0\tau} f(e^{ic\tau}\eta)e^{-i\hat{H}_0\tau} \quad (16)$$

to be solved in the perturbation scheme $\rho(\eta) = \sum_{n=0}^{\infty}\rho^{(n)}(\eta)$ using $\hat{V} \sim \lambda$ as a small parameter. Eq. (16) assumes the existence of the algebra of observables playing the roleof RDPs $\{\eta_a\}$, which obey the condition $[\hat{H}_0,\hat{\eta}_a] = \sum_b c_{ab}\hat{\eta}_b$ where c_{ab} are *C*-numbers, matrix $\mathbf{c}$ is present in (16). There also the operator function $f(\eta) = \left[\hat{V},\rho(\eta)\right] - i\sum_a \frac{\partial\rho(\eta)}{\partial\eta_a}L_a(\eta)$ is used [14]. Such method is known as Peletminskii–Yatsenko scheme [15]. The corresponding expansion for $L_a(\eta)$ provides constructing the kinetic equations for RDPs:

$$\partial_t\eta_a = L_a^{(0)}(\eta) + L_a^{(1)}(\eta) + L_a^{(2)}(\eta) + \ldots \quad (17)$$

where

$$L_a^{(0)}(\eta) = i\sum\nolimits_b c_{ab}\eta_b\ ,\ L_a^{(1)}(\eta) = i\mathrm{Sp}\rho^{(0)}(\eta)\left[\hat{V},\hat{\eta}_a\right],$$
$$L_a^{(2)}(\eta) = -\int_{-\infty}^{0} d\tau\,\mathrm{Sp}\rho^{(0)}(\eta)\left[\hat{V}(\tau),\left[\hat{V},\hat{\eta}_a\right] + i\sum\nolimits_b \hat{\eta}_b \frac{\partial L_a^{(1)}(\eta)}{\partial\eta_b}\right]. \qquad (18)$$

These terms are enough for our further calculations.

In Dicke model studies the most interesting and important for practical applications phenomenon is a superradiant pulse. The stored energy is emitted during a very short time. The pulse duration is many times shorter than the delay time – between the pumping irradiation light emission at the frequency of the operation transition. Electromagnetic power is liberated very quickly and leaves the Dicke system. Such a process is very complicated for description. Thus, it is expedient to fix this process via changes in the system of emitters. It is a system with much less degrees of freedom – "slow" system in Bogolyubov's terms. In emitter subsystem, a very fast change of the total difference of occupancy numbers of excited and ground level corresponds to the pulse emission. The most natural RDP in this problem is $\hat{R}_z = \sum_a \hat{R}_{az}$. Since for the Hamiltonian (12) $\hat{H}_0 = \hbar\Omega\sum_a \hat{R}_{az} + \sum_{\alpha,\mathbf{k}} \hbar\omega_k c_{\alpha\mathbf{k}}^{+} c_{\alpha\mathbf{k}}$ is the main part, the Peletminskii–Yatsenko condition $[\hat{H}_0, \hat{R}_z] = 0$ is satisfied trivially. If we are interested in correlations between emitters, the physical value with operator $\hat{R}^{+}\hat{R}^{-}$ should be considered (denotation $\hat{R}^{\pm} = \sum_a (\hat{R}_{ax} \pm i\hat{R}_{ay})$). Obviously $\hat{R}^{+}\hat{R}^{-} = \hat{R}_x^2 + \hat{R}_y^2 + \hat{R}_z$. For the concentrated model (with dimensions much less than the emitter wavelength) considered in the fundamental paper by Dicke [1] $\hat{R}_x^2 + \hat{R}_y^2 + \hat{R}_z^2$ is the motion integral, hence the maximum of the spontaneous emission reached at the moment of the maximal correlation coincides with the zero value of R_z. Notice that we can use $[\hat{H}_0, \hat{R}^{+}\hat{R}^{-}] = 0$, $[\hat{R}^{+}\hat{R}^{-}, \hat{R}_z] = 0$. Processes in a prolonged system, their spatial development and behavior of higher correlation functions require more thorough investigation.

The most complete information concerning an emitter subsystem is provided by its statistical operator. Such a program was put into life in our first papers on the Dicke model [15, 16]. For the Hamiltonian (12) ρ_m was chosen as RDP for the totality of emitters and the energy of photon bath E_f played this role for the field subsystem. The regular procedure of the RDM (14) – (18) gave the temporal equations of the described structure for RDPs:

$$\frac{\partial\rho_m(t,\rho_0)}{\partial t} = L_m\left(\rho_m(t,\rho_0), E_f(t,\rho_0)\right),$$
$$\frac{\partial E_f(t,\rho_0)}{\partial t} = L_f\left(\rho_m(t,\rho_0), E_f(t,\rho_0)\right). \qquad (19)$$

Then we dealt with L_m and L_f expansions of perturbation theory in interaction powers. Since the terms $L_m^{(1)}$ and $L_f^{(1)}$ proved to be zero, we obtained the substantiation that superradiance is a 2nd order effect. The fact that the term $L_f^{(2)} \sim V^0$ while E_f is proportional to V justifies the assumption $E_f = \mathrm{const}$. Hence we applied the concept of equilibrium photon thermostat at a

temperature *T*. In terms of emitter subsystem, the system evolution is described by the equation

$$\frac{\partial \rho_{\mathrm{m}}}{\partial t} = L_{\mathrm{m}}^{(2)}\left(\rho_{\mathrm{m}}(t,\rho_0), E_{\mathrm{f}}\right) = \frac{i}{\hbar}\frac{1}{V}\sum_k \int_{-\infty}^{0} \{\left[A_k^{\dagger}, \rho_{\mathrm{m}} A_k(\tau)\right] n_k - \\ -\left[A_k^{\dagger}, A_k(\tau)\rho_{\mathrm{m}}\right](1+n_k)\} e^{i\omega_k \tau} d\tau + h.c. \tag{20}$$

where $A_k(\tau)$ are atomic operators ininteraction presentation $A_k(\tau) = e^{\frac{i}{\hbar}\tau H_{\mathrm{m}}} A_k e^{-\frac{i}{\hbar}\tau H_{\mathrm{m}}}$, n_k are average photon occupation numbers $n_k = 1/\left(e^{\hbar\omega_k / k_{\mathrm{B}}T} - 1\right)$, and k_{B} is Boltzmann constant. *h.c.* denotes Hermitian conjugation here and further. In this presentation we made use of collective operators $A_k^{\dagger}$, A_k. Their view in the case of the concentrated model (12) is

$$A_k \equiv A_{\alpha,\mathbf{k}} = -2\sum_a \hat{R}_{ax} d\left(2\pi\hbar\omega_k\right)^{1/2}\left(\mathbf{n}_a \cdot \mathbf{e}_{\alpha}^{*}(\mathbf{k})\right), \\ A_k^{\dagger} \equiv A_{\alpha,\mathbf{k}}^{\dagger} = -2\sum_a \hat{R}_{ax} d\left(2\pi\hbar\omega_k\right)^{1/2}\left(\mathbf{n}_a \cdot \mathbf{e}_{\alpha}(\mathbf{k})\right). \tag{21}$$

Ignoring the problem of polarization and introducing an averaged interaction constant g_k, we come to $A_k = A_k^{\dagger} = \hbar g_k\left(R^{+} + R^{-}\right)$and summing over *k* in (20). At this stage it is possible to pass to prolonged systems, which are important in real experiments where the shape of the emitter subsystem determines the generated pulse direction. Account of the spatial localization of emitters is realized through introducing collective operators with corresponding factors: $R_{\mathbf{q}}^{\pm} = \sum_{a=1}^{N} R_a^{\pm} \exp\left(\pm i\mathbf{q}\cdot\mathbf{x}_a\right)$. Then atomic operators acquire the view

$$A_k = \hbar g_k \sum_{\mathbf{q}}\left(R_{\mathbf{q}}^{+} f^{*}\left(\mathbf{k}+\mathbf{q}\right) + R_{\mathbf{q}}^{-} f^{*}\left(\mathbf{k}-\mathbf{q}\right)\right) \tag{22}$$

using so called diffraction functions $f(\mathbf{k}) \equiv \frac{1}{N}\sum_a e^{i\mathbf{k}\cdot\mathbf{x}_a}$ [6, 16]. Thus, in the RDM framework the equation for emitter subsystem SO ρ_{m} ("Master equation") has been obtained [15]. At some assumptions (initial vacuum state of the field, excluding quickly oscillating terms) it coincides with the results of [6] where the physically adequate results were obtained for a "pencil-shape crystal".

RDM opens the way to investigating field behavior. The method of boson variable elimination [8] did not raise such questions. RDPs that should be involved in the scheme with this end in view are occupation numbers of possible field states and correlation functions of boson field under consideration. A special section of the review will be devoted to them.

4. Some new approaches and applications of RDM in superradiance theory

Constructing the "Master equation" can be implemented basing on the operation of calculating the trace of $\rho(t)$ in the space of thermostat states. Then the evolution

equation for $\rho_m(t)$ is presented in terms of Liouvilleans $\dot{\rho}_m(t) = \hat{L}_m \rho_m(t) + \mathrm{Sp}_f \hat{L}_{mf} \rho(t)$ and transformed to have a view

$$\frac{\partial \rho_m(t)}{\partial t} = -\frac{i}{\hbar}\left[\hat{H}_{ef}, \rho_m(t)\right] + \hat{L}_{dis}\rho_m(t). \tag{23}$$

Here $\hat{H}_{ef}$ – effective Hamiltonian of the emitter subsystem, $\hat{L}_{dis}$ – dissipative Liouvillean providing its irreversible evolution. In RDM, ρ is functionally dependent on ρ_m and E_f [16]. This determines the Liouville equation for $\rho(\rho_m, E_f)$, from which the integral equation follows for it using the condition of complete weakening of correlations [17]. Correspondingly, the right-hand side of (20) has no terms of 3rd order by interaction constant and its error is of the 4th order.

We obtained the picture of superradiance generation without some restrictions of [6], i.e. for an arbitrary boson thermostat temperature and account for antiresonant terms proportional to $f(\mathbf{k}+\mathbf{q})$. The kinetic equation for $\rho_m(t)$ includes double sums over $\mathbf{q}$ and $\mathbf{q}'$ (both belong to the first Brillouin zone *B*). It was shown in [16] that considering only $\mathbf{q}' = \pm\mathbf{q}$ is enough. The final view of operator structures in (23) is

$$\begin{aligned}
&\hat{H}_{ef} = \hbar(\Omega + \Delta\Omega)\hat{R}_0^z - \frac{1}{2}\sum_{\mathbf{q}\in B}\hbar\Omega_{\mathbf{q}}\hat{S}_{\mathbf{q}} + \frac{i}{2}\sum_{\mathbf{q}\in B}\left(O_{\mathbf{q}}\hat{R}_{\mathbf{q}}^+\hat{R}_{-\mathbf{q}}^+ - O_{\mathbf{q}}^*\hat{R}_{\mathbf{q}}^-\hat{R}_{-\mathbf{q}}^-\right),\\
&\hat{L}_{dis}\rho_m = \sum_{\mathbf{q}\in B}\Big\{\gamma_{\mathbf{q}}^-\left[R_{\mathbf{q}}^-, \rho_m R_{\mathbf{q}}^+\right] + \gamma_{\mathbf{q}}^+\left[R_{\mathbf{q}}^+, \rho_m R_{\mathbf{q}}^-\right] +\\
&\qquad +\delta_{\mathbf{q}}\left(R_{\mathbf{q}}^+\rho_m R_{-\mathbf{q}}^+ - 0.5\left[\rho_m, R_{\mathbf{q}}^+R_{-\mathbf{q}}^+\right]_+\right) + h.c.\Big\}
\end{aligned} \tag{24}$$

with the operator of cooperative polarization of emitters

$$\hat{S}_{\mathbf{q}} = \hat{R}_{\mathbf{q}}^+\hat{R}_{\mathbf{q}}^- - \left(\frac{1}{2}N + \hat{R}_{0z}\right) \tag{25}$$

and functions

$$\begin{aligned}
&\gamma_{\mathbf{q}}^+ = n(\Omega)\gamma_{\mathbf{q}},\ \gamma_{\mathbf{q}}^- = \left(1+n(\Omega)\right)\gamma_{\mathbf{q}},\ \gamma_{\mathbf{q}} \equiv \frac{1}{(2\pi)^2}\int d^3k g_k^2 f(\mathbf{k}-\mathbf{q})^2\,\delta(\omega_k - \Omega),\\
&\Omega_{\mathbf{q}} = \frac{1}{2\pi^3}P\int d^3k g_k^2\frac{\omega_k}{\omega_k^2 - \Omega^2}f^2(\mathbf{k}-\mathbf{q}),\quad \Delta\Omega = \frac{\Omega}{4\pi^3}P\int d^3k\frac{1+2n_k}{\Omega^2-\omega_k^2},\\
&O_{\mathbf{q}} = \frac{1}{2}\left(\gamma_{\mathbf{q}} + i\Omega_{\mathbf{q}}\right),\ \delta_{\mathbf{q}} = \gamma_{\mathbf{q}} + \frac{i\Omega}{2\pi^3}P\int d^3k\frac{g_k^2}{\omega_k^2-\Omega^2}f^2(\mathbf{k}-\mathbf{q})
\end{aligned} \tag{26}$$

using the designation $n(\Omega) \equiv 1/\left(e^{\hbar\Omega/k_B T} - 1\right)$.

One can see the presence of the frequency shift $\Delta\Omega$ of emitters connected with their interaction with field. Frequencies $\Omega_{\mathbf{q}}$ describe collective effects in the system and self-amplification of certain modes, such frequencies do not depend on the thermostat temperature.

In further such technique was applied to research into superradiant processes with taking into account additional factors: atom motions, higher correlation function etc. In [18], a generalized Dicke quasispin model of (12) type was considered, it was a system of two-level emitters forming a 3-dimensional lattice Q with ν particles in a site. The term H_{mf} was written in the usual form

$$\hat{H}_{\mathrm{mf}} = \frac{1}{\sqrt{V}} \sum_{\alpha,\mathbf{k}} \hbar g_{\alpha\mathbf{k}} \left(\hat{R}^{+}_{\mathbf{k}} c_{\alpha\mathbf{k}} + \hat{R}^{-}_{\mathbf{k}} c^{+}_{\alpha\mathbf{k}} \right), \tag{27}$$

but collective operators of emitters (Fourier transformed densities of particle quasispins) are modified:

$$\hat{R}^{\pm}_{\mathbf{k}} = \sum_{\substack{n\in Q \\ 1\le s\le \nu}} \hat{R}^{\pm}_{ns} e^{\pm i\mathbf{k}\cdot(\mathbf{x}_n + \hat{\mathbf{u}}_{ns})}, \qquad \hat{R}_{z\mathbf{k}} = \sum_{\substack{n\in Q \\ 1\le s\le \nu}} \hat{R}_{zns} e^{i\mathbf{k}\cdot(\mathbf{x}_n + \hat{\mathbf{u}}_{ns})} \tag{28}$$

where n numerates lattice sites and s numerates atoms in them. The displacement in site operators $\hat{\mathbf{u}}_{ns}$ are expressed via phonon operators $a_{\lambda\mathbf{q}}$, $a^{+}_{\lambda\mathbf{q}}$ for each atom of the lattice through the summation over the first zone of Brillouin B.

$$\hat{\mathbf{u}}_{ns} = \sum_{\substack{\mathbf{q}\in B \\ 1\le\lambda\le 3\nu}} \left(\frac{\hbar}{2Nm_s\omega_\lambda(\mathbf{q})} \right)^{1/2} \left\{ \mathbf{e}_{\lambda s}(\mathbf{q}) a_{\lambda\mathbf{q}} e^{i\mathbf{q}\cdot\mathbf{x}_n} + h.c. \right\}. \tag{29}$$

The system Hamiltonian should be supplemented with the term corresponding to phonons $\hat{H}_{\mathrm{p}} = \sum_{\lambda,\mathbf{q}} \hbar\omega_\lambda(\mathbf{q}) a^{+}_{\lambda\mathbf{q}} a_{\lambda\mathbf{q}}$.

At the first stage of RDM, we chose the statistical operator of emitter subsystem ρ_{m} as well as energies of photon and phonon subsystems E_{f} and E_{p} to be used as RDPs for our problem. Proceeding from the basic kinetic equation in the form (23) and the assumption of equilibrium boson thermostats, we came to the effective Hamiltonian and dissipative term for the emitter subsystem and evolution equations for boson subsystems

$$\begin{aligned}
\hat{H}_{\mathrm{ef}} &= \sum_{n,s} \hbar\Omega \hat{R}_{zns} + \sum_{n,s,n',s'} \left(A_{ns,n's'} \hat{R}^{+}_{ns} \hat{R}^{-}_{n's'} + B_{ns,n's'} \hat{R}^{-}_{ns} \hat{R}^{+}_{n's'} \right) + O\left(g^4\right), \\
\hat{L}_{\mathrm{dis}}\rho_{\mathrm{m}} &= \left[\sum_{n,s,n',s'} \left(C_{ns,n's'} \hat{R}^{+}_{ns} \hat{R}^{-}_{n's'} + D_{ns,n's'} \hat{R}^{-}_{ns} \hat{R}^{+}_{n's'} \right), \rho_{\mathrm{m}} \right]_{+} + O\left(g^4\right), \\
\partial_t E_{\mathrm{f}} &= \sum_{n,s,n',s'} \mathrm{Sp}_{\mathrm{m}} \rho_{\mathrm{m}} \left(G_{ns,n's'} \hat{R}^{+}_{ns} \hat{R}^{-}_{n's'} + H_{ns,n's'} \hat{R}^{-}_{ns} \hat{R}^{+}_{n's'} \right) + O\left(g^4\right), \\
\partial_t E_{\mathrm{p}} &= \sum_{n,s,n',s'} \mathrm{Sp}_{\mathrm{m}} \rho_{\mathrm{m}} \left(M_{ns,n's'} \hat{R}^{+}_{ns} \hat{R}^{-}_{n's'} + N_{ns,n's'} \hat{R}^{-}_{ns} \hat{R}^{+}_{n's'} \right) + O\left(g^4\right).
\end{aligned} \tag{30}$$

The Hermitian matrixes $A_{ns,n's'}$,..., $N_{ns,n's'}$ depend on time through $E_{\mathrm{f}}(t)$ and $E_{\mathrm{p}}(t)$. They are expressed via the Planck distribution $n_k = \left(e^{\frac{\hbar\omega_k}{k_{\mathrm{B}}T}} - 1 \right)^{-1}$ and van Hove type functions [19]

$$g_{ns,n's'}(\mathbf{k},\tau) \equiv \mathrm{Sp}_{\mathrm{p}} w_{\mathrm{p}} e^{i\mathbf{k}\cdot(\mathbf{x}_n + \hat{\mathbf{u}}_{ns})} e^{-i\mathbf{k}\cdot(\mathbf{x}_n + \hat{\mathbf{u}}_{ns}(\tau))} \tag{31}$$

with w_{p} standing for the equilibrium statistical operator of phonons.

If we study the Dicke process in a crystal, the conventional approach is investigating the behavior of the average value of $\hat{R} \equiv \sum_{n,s} \hat{R}_{zns}$. Regarding the set of RDPs $R(t), E_{\mathrm{f}}(t), E_{\mathrm{p}}(t)$ (we shall use for boson subsystem parameters the general notation η) and using the constructed "Master equation" $\dot{\rho}_{\mathrm{m}}(t) = \left\{\hat{L}_0 + \hat{L}_{\mathrm{int}}(\eta(t)) + \hat{L}_{\mathrm{dis}}(\eta(t))\right\}\rho_{\mathrm{m}}(t)$, we obtain the set of differential equations

$$\dot{R}(t) = L\left(R(t), \eta(t)\right), \quad \dot{\eta}_i = \tilde{L}_i\left(R(t), \eta(t)\right), \tag{32}$$

which will be referred to as generalized Rehler-Eberly equations. Here $L(R,\eta) = \mathrm{Sp}_{\mathrm{m}}\hat{R}\left(\hat{L}_{\mathrm{int}}(\eta) + \hat{L}_{\mathrm{dis}}(\eta)\right)\rho_{\mathrm{m}}(R,\eta)$, $\tilde{L}_i(R,\eta) \equiv \hat{L}_i\left(\rho_{\mathrm{m}}(R,\eta),\eta\right)$. In RDM scheme right-hand sides of (32) acquire the view

$$L = \mathrm{Sp}_{\mathrm{m}} R\left(\hat{L}^{(2)}_{\mathrm{int}}, \hat{L}^{(2)}_{\mathrm{dis}}\right)\rho_{\mathrm{q}} + O\left(g^4\right), \quad \tilde{L}_i = \hat{L}^{(2)}_i\left(\rho_{\mathrm{q}},\eta\right) + O\left(g^4\right) \tag{33}$$

via the iteration procedure for ρ_{m} with a quasi-equilibrium SO ρ_{q} used as a zero-th approximation. The problem of calculating averages with it will be considered later.

Aiming to explore correlations between emitters, we come to studying the "Master equation" of (23) type with

$$\begin{aligned} \hat{H}_{\mathrm{ef}} &= \sum_n \hbar\Omega\hat{R}_{zn} + \sum_{n,n'}\left(I_{nn'}\hat{R}_{zn}\hat{R}_{zn'} + \tilde{I}_{nn'}\left(R^+_n\hat{R}^-_{n'} + \hat{R}^-_n\hat{R}^+_{n'}\right)\right) \equiv \hat{H}_0 + \hat{H}_{\mathrm{int}}, \\ \hat{L}_{\mathrm{dis}}\rho_{\mathrm{m}} &= \left[\sum_{n,n'}\left(A_{nn'}\hat{R}_{zn}\hat{R}_{zn'} + B_{nn'}\left(\hat{R}^+_n\hat{R}^-_{n'} + \hat{R}^-_n\hat{R}^+_{n'}\right)\right), \rho_{\mathrm{m}}\right]_+ + \sum_{n,n'} C_{nn'}\left(\hat{R}^+_n\rho_{\mathrm{m}}\hat{R}^-_{n'} + \hat{R}^-_n\rho_{\mathrm{m}}\hat{R}^+_{n'}\right). \end{aligned} \tag{34}$$

Expression for the effective Hamilton operator $\hat{H}_{\mathrm{ef}}$ is close to the theory of an anisotropic Heisenberg magnetic [20]. Our assumptions are: $\hat{H}_{\mathrm{int}} \sim \lambda$, $\hat{L}_{\mathrm{dis}} \sim \lambda$, $\lambda << 1$ is a small parameter. We study the stage of emitter system evolution, at which it can be described by RDPs

$$R^{\pm}_n(t) = \mathrm{Sp}\rho_{\mathrm{m}}(t)\hat{R}^{\pm}_n. \tag{35}$$

Averaged SO of the system can be introduced too, that gives [21]

$$\tilde{\rho}_{\mathrm{m}}\left(\tilde{R}(t), \tilde{g}(t)\right) \equiv \overline{\rho_{\mathrm{m}}\left(R(t)\right)}, \qquad \tilde{\rho}_{\mathrm{m}}\left(\tilde{R}, \tilde{g}\right) = e^{G\left(\tilde{g}, \frac{\partial}{\partial R}\right)}\rho_{\mathrm{m}}(R)_{R\to\tilde{R}}. \tag{43}$$

It was shown in [21] that equations (41) coincide with (39) and $\tilde{\rho}_{\mathrm{m}}(R,g) = \rho_{\mathrm{m}}(R,g)$ if the projecting operator $P_{nn'}$ in (38) destroys short-range emitter-emitter correlations. So, it is possible on the basis of equations (41) to investigate kinetics of emitters taking into account long-range nonequilibrium fluctuation (correlations).

According to ideas developed in [20] for spin systems, the 1st of equations (41) describes mean quasispin field evolution. The 2nd of equations (41) describes quasispin field excitations. In this sense one can consider $\tilde{g}^{\sigma''\sigma'}_{n''n'}$ as a density matrix of excitations. For our system of emitters these excitations can be named the Frenkel

excitons. Thus, kinetic equations for excitons in a usual sense have been obtained. The ideas of Section 4 were summarized in our paper [22].

5. Reduced description of electromagnetic field in Dicke process

As we have pointed out above, the possibility of studying electromagnetic field states is an important advantage of reduced description method. In general, a quantum field state can be presented by the totality of occupation numbers of field modes, but such information is redundant. Physically valuable information is obtained through using such field characteristics as electric and magnetic component strengths. In quantum theory, operators dependent on space and time coordinates correspond to these variables. Their average (in quantum sense) values give the picture of field evolution. More precise information about a process is presented by correlation functions of different orders of specified variables.

Thus, Dicke Hamiltonian (12) should be rewritten using relevant field variables. Since publishing [23], we put down the Hamiltonian under consideration appealing to notions of electrodynamics of continuous media:

$$\hat{H}_{\mathrm{mf}} = -\int d\mathbf{x}\hat{\mathbf{E}}^{t}(\mathbf{x})\cdot\hat{\mathbf{P}}(\mathbf{x}). \tag{44}$$

Here $\hat{\mathbf{P}}(\mathbf{x}) \equiv \sum_{a} 2\hat{R}_{ax}\mathbf{d}_{a}\delta(\mathbf{x}-\mathbf{x}_{a})$ is the operator of electric dipole moment density of emitters, the spatial orientation of $\mathbf{d}_{a}$ is supposed to be uniform. The nonequilibrium state of the field will be described by parameters $\{\eta\}$ that include the average transverse electric and magnetic field strengths ζ_{μ} with the operators

$$\begin{aligned}
\hat{E}_{n}^{t}(\mathbf{x}) &= \sum_{\mathbf{k},\alpha}\left(\frac{2\pi\hbar\omega_{k}}{V}\right)^{1/2} e_{\alpha n}(\mathbf{k})\left(c_{\alpha\mathbf{k}} - c_{\alpha,-\mathbf{k}}^{+}\right)e^{-i\mathbf{k}\cdot\mathbf{x}},\\
\hat{B}_{n}(\mathbf{x}) &= \sum_{\mathbf{k},\alpha}\left(\frac{2\pi\hbar\omega_{k}}{V}\right)^{1/2} \left(\tilde{\mathbf{k}}\times e_{\alpha n}(\mathbf{k})\right)\left(c_{\alpha\mathbf{k}} + c_{\alpha,-\mathbf{k}}^{+}\right)e^{i\mathbf{k}\cdot\mathbf{x}}
\end{aligned} \tag{45}$$

and their binary correlations $(\zeta_{\mu},\zeta_{\mu'})$. Here μ and μ′ numerate new RDPs, n is a vector index, $\tilde{\mathbf{k}} = \mathbf{k}/k$, $e_{\alpha n}(\mathbf{k})$ corresponds to the linear polarization of modes. The general definition of the binary correlation function of operators $\hat{a}$ and $\hat{b}$ is given by the formula with the statistical operator ρ of the system and their anticommutator

$$(a,b) = \mathrm{Sp}\,\rho\{\hat{a},\hat{b}\}/2 - \mathrm{Sp}\,\rho\hat{a}\,\mathrm{Sp}\,\rho\hat{b}. \tag{46}$$

The state of the medium will be described by the average density $\varepsilon(\mathbf{x})$ of its energy

$$\varepsilon(\mathbf{x}) = \mathrm{Sp}\,\rho\hat{\varepsilon}(\mathbf{x}), \quad \hat{\varepsilon}(\mathbf{x}) = \sum_{a}\hbar\Omega\hat{R}_{az}\delta(\mathbf{x}-\mathbf{x}_{a}). \tag{47}$$

Introducing the velocity change operator of the quantity with the operator $\hat{a}$ in Schrödinger picture: $\dot{\hat{a}} \equiv \frac{i}{\hbar}[\hat{H},\hat{a}]$, we come to Maxwell's operator equations

$$\dot{\hat{\mathbf{B}}} = -c\,\mathrm{rot}\hat{\mathbf{E}}, \quad \dot{\hat{\mathbf{E}}} = c\,\mathrm{rot}_n\hat{\mathbf{B}} - 4\pi\hat{\mathbf{I}}, \quad \mathrm{div}\hat{\mathbf{B}} = 0, \quad \mathrm{div}\hat{\mathbf{E}} = 4\pi\hat{\rho} \tag{48}$$

using the operators of complete electric field $\hat{\mathbf{E}} = \hat{\mathbf{E}}^t - 4\pi\hat{\mathbf{P}}$ and current $\hat{\mathbf{I}} \equiv \dot{\hat{\mathbf{P}}}$ and charge $\hat{\rho} \equiv -\mathrm{div}\hat{\mathbf{P}}$ density. The operator equation for medium energy density takes the form

$$\hat{\varepsilon}(\mathbf{x}) = \hat{\mathbf{I}}(\mathbf{x}) \cdot \hat{\mathbf{E}}^t(\mathbf{x}). \tag{49}$$

In [24], the compact form of evolution equations for averages of field RDPs was substantiated

$$\partial_t \zeta_\mu = i\sum_{\mu'} \mathbf{c}_{\mu\mu'}\zeta_{\mu'} + Q_\mu, \qquad Q_\mu : 0,\ 4\pi c\,\mathrm{rot}\mathbf{P} \tag{50}$$

where $\mathbf{c}_{\mu\mu'}$ issome numerical matrix. The equation (50) results in the evolution equation for the binary correlation functions for field variables

$$\partial_t(\zeta_{\mu_1}, \zeta_{\mu_2}) = i\sum_{\mu_1'} \mathbf{c}_{\mu_1\mu_1'}(\zeta_{\mu_1'}, \zeta_{\mu_2}) + i\sum_{\mu_2'} \mathbf{c}_{\mu_2\mu_2'}(\zeta_1, \zeta_{2'}) + (Q_{\mu_1}, \zeta_{\mu_2}) + (\zeta_{\mu_1}, Q_{\mu_2}). \tag{51}$$

Equations (50) and (51) give a complete set of temporal equations for the RDPs of the electromagnetic field in the medium (we will denote all of them as $\{\eta_a\}$). Together with the evolution equation for $\varepsilon(\mathbf{x})$, they give the whole picture of the Dicke process taking into account binary correlations in electromagnetic field. In this case Bogolyubov functional hypothesis view is $\rho(t) \xrightarrow[t>>\tau_0]{} \rho(\eta(t), \varepsilon(t))$ with spatially dependent field and emitter variables. Let`s remember that in Dicke model $\hat{H}_0 = \hat{H}_\mathrm{m} + \hat{H}_\mathrm{f}$, hence the Peletminskii – Yatsenko requirements are required (see (14) – (18)). The non-diagonal matrix element *d* of the dipole moment operator of an emitter plays the small parameter role.

In the approach under consideration, the quasi-equilibrium operator of Eq. (16) takes the form

$$\rho_\mathrm{q}(\eta, \varepsilon) = \rho_\mathrm{f}(\eta)\rho_\mathrm{m}(\varepsilon). \tag{52}$$

$\rho_\mathrm{f}(\eta)$ is aquasi-equilibrium statistical operator of electromagnetic field, its exponent indicator includes a quadratic form of Bose operators providing the existence of traces. $\rho_\mathrm{m}(\varepsilon)$ is a product of the exponent with a quasispin operator $\hat{\varepsilon}(\mathbf{x})$ (47) in the indicator and the uniform angular distribution of atom dipole moments, it depends on local medium temperature. It was shown [24] that 1st order (in *d*) contributions to the kinetic equations (see (17)) for RDPs η and ε are zero as well as $\mathbf{P}^{(1)}$ and $\mathbf{I}^{(1)}$. That is why the 1st order contribution to the SO (52) is reduced to

$$\rho^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{0} d\tau \int d\mathbf{x}\left[\rho_\mathrm{f}\rho_\mathrm{m}, \hat{\mathbf{E}}^t(\mathbf{x}, \tau) \cdot \hat{\mathbf{P}}(\mathbf{x}, \tau)\right] \tag{53}$$

where the time dependence of functions means

$$\hat{\mathbf{E}}^t(\mathbf{x}, \tau) \equiv e^{\frac{i}{\hbar}\tau\hat{H}_\mathrm{f}}\hat{\mathbf{E}}^t(\mathbf{x})e^{-\frac{i}{\hbar}\tau\hat{H}_\mathrm{f}}, \ \hat{\mathbf{P}}(\mathbf{x}, \tau) \equiv e^{\frac{i}{\hbar}\tau\hat{H}_\mathrm{m}}\hat{\mathbf{P}}(\mathbf{x})e^{-\frac{i}{\hbar}\tau\hat{H}_\mathrm{m}}. \tag{54}$$

Drawing on Peletminskii – Yatsenko conditions, we obtain from this the time dependence of $\hat{\mathbf{P}}(\mathbf{x},\tau)$ and field Fourier component $\hat{\mathbf{E}}_k^t$, the last depending on ω_k. On this basis it is possible to calculate the 2nd order term of average polarization

$$P_n^{(2)}(\mathbf{x}) = \frac{i}{\hbar}\int_{-\infty}^{0} d\tau \int d\mathbf{x}' \,\mathrm{Sp}_\mathrm{f}\,\rho_\mathrm{f}\hat{E}_l^t(\mathbf{x}',\tau)\,\mathrm{Sp}_\mathrm{m}\,\rho_\mathrm{m}\left[\hat{P}_n(\mathbf{x}),\hat{P}_l(\mathbf{x}',\tau)\right] \tag{55}$$

where $\left[\hat{P}_n(\mathbf{x}),\hat{P}_l(\mathbf{x}',\tau)\right] = -4i\delta(\mathbf{x}-\mathbf{x}')\hat{a}_{nl}(\mathbf{x})\sin\Omega\tau$, $\hat{a}_{nl}(\mathbf{x}) \equiv \sum_a d_{an}d_{al}\hat{R}_{az}\delta(\mathbf{x}-\mathbf{x}_a)$. In terms of Fourier images, we have

$$P_n^{(2)}(\mathbf{x}) = \frac{4}{\hbar V}\sum_k e^{i\mathbf{k}\cdot\mathbf{x}}\int_{-\infty}^{0} d\tau\left(E_{lk}^t\cos\omega_k\tau + cZ_{lk}\omega_k^{-1}\sin\omega_k\tau\right)a_{nl}(\mathbf{x})\sin\Omega\tau \tag{56}$$

where $a_{nl}(\mathbf{x}) \equiv \mathrm{Sp}_\mathrm{m}\,\rho_\mathrm{m}\hat{a}_{nl}(\mathbf{x})$, $\mathrm{Sp}_\mathrm{f}\,\rho_\mathrm{f}\hat{E}_{lk}^t = E_{lk}^t$, $\mathrm{Sp}_\mathrm{f}\,\rho_\mathrm{f}\hat{Z}_{lk} = Z_{lk}$, $Z_l(\mathbf{x}) \equiv \mathrm{rot}_l\,\mathbf{B}(\mathbf{x})$. Averaging over directions gives connection with medium energy density for the function $a_{nl}(\mathbf{x})$:

$$a_{nl}(\mathbf{x}) = \frac{d^2}{3\hbar\Omega}\varepsilon(\mathbf{x})\delta_{nl}\,. \tag{57}$$

Integrating over time in generalized functions results in the material equation of electrodynamics in the case of a spatially inhomogeneous medium:

$$P_n^{(2)}(\mathbf{x}) = \frac{1}{V}\sum_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}[\kappa(\mathbf{k},\varepsilon(\mathbf{x}))E_{n\mathbf{k}}^t + c\lambda(\mathbf{k},\varepsilon(\mathbf{x}))Z_{n\mathbf{k}}] \tag{58}$$

with designations

$$\kappa(\mathbf{k},\varepsilon) = -\frac{4d^2}{3\Omega\hbar^2}\varepsilon\mathrm{P}\frac{\Omega}{\Omega^2-\omega_k^2}\,, \qquad \lambda(\mathbf{k},\varepsilon) = \frac{2\pi d^2}{3\Omega^2\hbar^2}\varepsilon\delta(\Omega-\omega_k)\,. \tag{59}$$

In view of $\partial_t\varepsilon \sim d^2$, we should not differentiate ε when calculating current density and therefore

$$\begin{aligned} I_n^{(2)}(\mathbf{x}) &= \frac{1}{V}\sum_k e^{i\mathbf{k}\cdot\mathbf{x}}[\sigma(\mathbf{k},\varepsilon(\mathbf{x}))E_{n\mathbf{k}}^t + c\xi(\mathbf{k},\varepsilon(\mathbf{x}))Z_{n\mathbf{k}}]\,, \\ I_n^{(2)}(\mathbf{x}) &= \int d\mathbf{x}'[\sigma(\mathbf{x}-\mathbf{x}',\varepsilon(\mathbf{x}))E_{n\mathbf{k}}^t + c\xi(\mathbf{x}-\mathbf{x}',\varepsilon(\mathbf{x}))Z_{n\mathbf{k}}] \end{aligned} \tag{61}$$

where

$$\begin{aligned} \sigma(\mathbf{k},\varepsilon) &\equiv -\lambda(\mathbf{k},\varepsilon)\omega_k^2 = -\frac{2\pi d^2}{3\hbar^2}\varepsilon\,\delta(\Omega-\omega_k)\,, \\ \xi(\mathbf{k},\varepsilon) &\equiv \kappa(\mathbf{k},\varepsilon) = -\frac{4d^2}{3\hbar^2}\varepsilon\,\mathrm{P}\frac{1}{\Omega^2-\omega_k^2} \end{aligned} \tag{62}$$

Notice that coefficients κ and σ characterize dielectric susceptibility and conductivity, respectively, coefficients λ and ξ relate spatial inhomogeneity and frequency

dispersion. Obviously, averaged charge density in the medium is zero in the case of spatial homogeneity.

Thus, we have constructed electrodynamics of a medium of two-level emitters. Its equations are

$$\partial_t \mathbf{B} = -c\,\mathrm{rot}\mathbf{E}\,, \quad \partial_t \mathbf{E} = c\,\mathrm{rot}\mathbf{B} - 4\pi\mathbf{I}\,, \quad \mathrm{div}\mathbf{B} = 0\,, \quad \mathrm{div}\mathbf{E} = 4\pi\rho\,. \tag{63}$$

Equations (59) and (62) provide material equations for them. The set (63) is of usual form. Since P_n is proportional to d^2, a complete field in (63) can be replaced with the transversal one. Accepted RDPs are $\mathbf{E}^t$, $\mathbf{B}$ and their binary correlations. but all obtained equations are valid with considered accuracy if we use the complete field for the process description. At the same time, material equations do not include binary correlations.

Putting forward the task to study the medium energy density evolution, we face the problem of solving the differential equation for ε (49) using in its right-hand side $L^{(2)}(x)$ obtained results concerning $\mathbf{P}$ and $\mathbf{I}$. Rather complicated calculations give the expression[24]

$$\partial_t \varepsilon(\mathbf{x}) = \sum_n \left\{ (I_n(\mathbf{x}), E_n(\mathbf{x})) + I_n(\mathbf{x}) E_n(\mathbf{x}) \right\} - \frac{2d^2\Omega^4}{3c^3} n(\mathbf{x}) + O(d^3)\,. \tag{64}$$

Besides the correlation functions (46) between $\mathbf{I}$ and $\mathbf{E}$ components and their scalar product, Eq. (64) contains the term describing system energy reduction due to dipole radiation, which is proportional to the spatial density of emitters $n(\mathbf{x}) = \dfrac{\varepsilon(\mathbf{x})}{\hbar\Omega}$.

6. Correlation functions in RDM

Pay attention to the fact that our technique provides the possibility of studying the behavior of correlation functions in Dicke process. Such functions play a decisive role in quantum optics research. Conventional approach to superfluorescencepresumes considering the time of pulse delay and the total quasispin evolution. Thus, the process of correlation development is an open question. In early paper [25], the superradiance picture with higher correlations in emitter subsystem was built in the framework of the method of boson variable elimination for a concentrated Dick model. This methodgenerates the chainof linked equations involvingmore and more complicated functions. The result of [25] was very simple: higher correlation function account does not influence the delay time. At the same time numerical experiments demonstrate the exponential growth of correlation functions. An analogous problem for a prolonged system has no transparent solution. But RDM gives us a chance to simulate the development of correlations of field parameters in an arbitrary system. With this end in view, we considered the set of equations describing the evolution of binary field correlation function.The choice of RDPs is unlimited though technical difficulties grow very quickly.

In [26], we tried to construct an evolution equation for a binary correlation function (50) made of strength parameters of electromagnetic field. Proceeding from the compact operator equation form of (50) type

$$\hat{\dot{\xi}}_1 = i\sum_2 \mathbf{c}_{12}\hat{\xi}_2 - 4\pi\hat{I}_1 \tag{65}$$

where

$$\hat{\xi}_{1n}(\mathbf{x}) = \hat{E}_n(\mathbf{x}), \quad \hat{\xi}_{2n}(\mathbf{x}) = \hat{B}_n(\mathbf{x}); \qquad I_{1n}(\mathbf{x}) = 0, \qquad \hat{I}_{2n}(\mathbf{x}) = \hat{I}_n(\mathbf{x}) \tag{66}$$

(n is a vector index, matrix $\mathbf{c}_{12}$ coincides with the corresponding one in (50)), we write down the equation for a correlation function as

$$\partial_t(\xi_1,\xi_2) = i\sum_{1'}\mathbf{c}_{11'}(\xi_{1'},\xi_2) + i\sum_{2'}\mathbf{c}_{22'}(\xi_1,\xi_{2'}) - 4\pi(I_1,\xi_2) - 4\pi(\xi_1,I_2). \tag{67}$$

In usual notations, the corresponding equations are given by the formulas

$$\begin{aligned}
\partial_t(B_n^x B_l^{x'}) &= -c\,\mathrm{rot}_n(E^x B_l^{x'}) - c\,\mathrm{rot}'_l(B_n^x E^{x'}),\\
\partial_t(B_n^x E_l^{x'}) &= -c\,\mathrm{rot}_n(E^x E_l^{x'}) + c\,\mathrm{rot}'_l(B_n^x B^{x'}) - 4\pi(B_n^x I_l^{x'}),\\
\partial_t(E_n^x E_l^{x'}) &= c\,\mathrm{rot}_n(B^x E_l^{x'}) + c\,\mathrm{rot}'_l(E_n^x B^{x'}) - 4\pi(I_n^x E_l^{x'}) - 4\pi(E_n^x I_l^{x'})
\end{aligned} \tag{68}$$

(hereafter for the simplicity $(E_n(\mathbf{x}), B_l(\mathbf{x}')) \equiv (E_n^x B_l^{x'})$ and so on).

To find the material equations for evolution equations for correlations one must calculate nonequilimrium correlation functions $(B_n^x I_l^{x'})$, $(E_n^x I_l^{x'})$. The correlations are found in the main (second) approximation in the field-medium interaction, i.e. in dipole moment of an emitter d). So, it is enough considering the correlation function $(E_n^{tx} I_l^{x'})$.The necessary calculations are like those conducted for obtaining (64). Let's give the results [26].

$$\begin{aligned}
(E_n^{tx} I_l^{x'})^{(2)} &= (E_n^{tx} I_l^{(2)x'}) + S_{nl}(\mathbf{x}-\mathbf{x}', n(\mathbf{x}')),\\
S_{nl}(\mathbf{x},n) &= -n\frac{d^2\omega^2}{12\pi^2}\int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}\delta^t_{nl}\delta(\Omega-\omega_k)
\end{aligned} \tag{69}$$

$$\begin{aligned}
(B_n^x I_l^{x'})^{(2)} &= (B_n^x I_l^{(2)x'}) + T_{nl}((\mathbf{x}-\mathbf{x}', n(\mathbf{x}'))),\\
T_{nl}(\mathbf{x},n) &\equiv n\frac{cd^2}{3} e_{nls}\frac{\partial}{\partial x_s}\frac{1}{2\pi^2}\int d^3\mathbf{k} e^{i\mathbf{k}\cdot\mathbf{x}}\,\mathrm{P}\frac{\Omega^2}{\Omega^2-\omega_k^2}.
\end{aligned} \tag{70}$$

The presence of the functions $S_{nl}(\mathbf{x},n)$ and $T_{nl}(\mathbf{x},n)$ in (69) and (70) is an interesting fact.Their expressions include transversal δ-symbol $\delta^t_{nl} \equiv \delta_{nl} - k_n k_l / k^2$ and Levi-Civita tensor ε_{nlm}. These formulas are necessary supplementsto the equations (67). It is worthto emphasize that material equations for correlations do not contain in the considered approximation average electromagnetic field. On the other hand, the material equations (58) and (61) to Maxwell equations do not contain binary correlations. However, due to the specified terms in (69) and (70), these material equations depend on the density of medium energy. According to evolution equation for the energy density (64), it is changed under the influenceboth the field and its

correlations. So, in the case of nonequilibrium medium evolution of the field and correlations are connected. Thus, we have built electrodynamics of a continuous medium formed by two-level emitters (with fixed positions)taking into account binary correlation functionswith accuracy up to 2nd order terms.

The obtained theory opens the way to the numerical analysis of field correlation development in Dicke system, though expected calculations are very complicated. Such an attempt was done in paper [27]. Solving the Cauchy problem for the equations of (68) type requires material equations in a spatial form (see 2nd equation of (92), the upper index of I_n there may be omitted). Due to the representation

$$\sigma(\mathbf{x}-\mathbf{x}',\varepsilon(\mathbf{x}))=\frac{1}{V}\sum_{\mathbf{k}}\sigma(\mathbf{k},\varepsilon(\mathbf{x}))e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')}\,,\ \xi(\mathbf{x}-\mathbf{x}',\varepsilon(\mathbf{x}))=\frac{1}{V}\sum_{\mathbf{k}}\xi(\mathbf{k},\varepsilon(\mathbf{x}))e^{i\mathbf{k}\cdot(\mathbf{x}-\mathbf{x}')} \qquad (71)$$

and the transformation that is valid in the thermodynamical limit $\sum_{\mathbf{k}}...\overset{TL}{=}\frac{V}{(2\pi)^3}\int d^3\mathbf{k}...$ we can obtain the necessary values via integrating in spherical coordinates.If we neglect the Lorentz broadening, in integrals over $k=|\mathbf{k}|$ such functions as $\delta(k-\Omega/c)$ appear.The results are

$$\begin{gathered}\sigma\left(\mathbf{x}-\mathbf{x}',\varepsilon(\mathbf{x})\right)=-\frac{d^2}{3\pi c\hbar^2}\varepsilon(\mathbf{x})\frac{\sin\left(\Omega|\mathbf{x}-\mathbf{x}'|/c\right)}{\Omega|\mathbf{x}-\mathbf{x}'|/c},\\ \xi(\mathbf{x}-\mathbf{x}',\varepsilon(\mathbf{x}))=-\frac{2d^2}{3\pi\hbar^2\Omega^2}\frac{\varepsilon(\mathbf{x})}{|\mathbf{x}-\mathbf{x}'|}(1-\cos(\Omega|\mathbf{x}-\mathbf{x}'|/c)\,.\end{gathered} \qquad (72)$$

In spite ofthe rather simple structure of equations (63), (64), (68) and the availability of expressions (72), the set of kinetic equations remains very difficult for simulations because of the great number of variables, presence of spatial differentiation and integration as well as additional terms in (69) and (70). A natural way of simplification is considering a system of a smaller number of dimensions. In [27], a one-dimensional (along axis *z*) crystal was proposed for consideration. Dipole moment orientation was also fixed (such supposition may be physically substantiated), field amplitude vectors $\mathbf{E}\|\mathrm{O}x$, $\mathbf{B}\|\mathrm{O}y$ and wave vector $\mathbf{k}\|\mathrm{O}z$. Thus, we came to the set of field RDPs $E\equiv E_x(z)$, $B\equiv B_y(z)$, $(EE')\equiv(E_x(z),E_x(z'))$, $(EB')\equiv(E_x(z),B_y(z'))$, $(BB')\equiv(B_y(z),B_y(z'))$. Kinetic equations for them are

$$\begin{gathered}\partial_t E=-c\frac{\partial B}{\partial z}-4\pi I(z),\qquad \partial_t B=c\frac{\partial E}{\partial z},\\ \partial_t(EE')=-c\frac{\partial(E'B)}{\partial z}-c\frac{\partial(EB')}{\partial z'}-4\pi(IE')-4\pi(EI'),\\ \partial_t(BB')=-c\frac{\partial(EB')}{\partial z}+c\frac{\partial(BE')}{\partial z'},\ \partial_t(EB')=-c\frac{\partial(BB')}{\partial z}+c\frac{\partial(EE')}{\partial z'}-4\pi(IB')\,.\end{gathered} \qquad (73)$$

At the first stage of the Dicke process, we may neglect the changes of the emitter subsystem energy density and take the fixed value of this parameter. In [28],

advanced version of (73) was constructed for a two-dimensional system. Real computer simulations are in our plans.

Another interesting result of the superradiance field behavior analysis with using the ideas of electrodynamics of continuous media, which are applicable just at the beginning of Dicke arranging, is dispersion relations for waves in Dicke model [29]. We used the material equations (62), but considered also the non-uniform broadening, i.e. the coefficients were taken in the form

$$\sigma(\mathbf{k},\varepsilon) = -s\int_0^{+\infty} d\omega\, w_\alpha(\omega)\delta(\omega-\omega_k)\,,\ \xi(\mathbf{k},\varepsilon) = -\frac{2s}{\pi}\int_0^{+\infty} d\omega\, w_\alpha(\omega)\mathrm{P}\frac{1}{\omega^2-\omega_k^2} \tag{74}$$

where $\omega_k = ck$, $s \equiv 2\pi d^2\varepsilon/3\hbar^2$ and, in accordance with [?]

$$w_\alpha(\omega) = c_\alpha \frac{\alpha}{(\omega-\Omega)^2+\alpha^2}\,,\ \int_0^{+\infty} d\omega w_\alpha(\omega) \equiv 1 \tag{75}$$

Standard electrodynamics procedure for such medium gives the specified equation in the view

$$\left|\left(k^2 - \frac{\omega^2}{c^2}u(\mathbf{k},\omega)\right)\delta_{nl} - k_n k_l\right| = 0 \tag{76}$$

with $u(\mathbf{k},\omega) \equiv \dfrac{1+4\pi i\sigma(\mathbf{k},\varepsilon)\omega^{-1}}{1-4\pi\xi(\mathbf{k},\varepsilon)}$. We suppose an isotropic case $u(\mathbf{k},\omega) = u(k,\omega)$, so the vector $\mathbf{E}$ structure is defined only by the vector $\mathbf{k}$ and can have the longitudinal $\mathbf{E}^l$ and transversal $\mathbf{E}^t$ components. Searching for dispersion laws for them in the form $\omega(k) \equiv \omega' + i\omega''$ shows:for longitudinal waves propagation is impossible, but the positive sign of ω'' evidences that field energy grows rapidly when ck is close to Ω; for transversal waves $\omega'' = 2\pi s w_\alpha(ck)$ and ω' gives the possibility of calculating their group velocity, which value decrease relative to c is proportional to the small parameter s. All results are physically convincing and provide the picture of superradiance in classical terms.

Obviously, correlations in field states are coordinated with some correlations between emitters. This connection description is an interesting problem for further investigations. As we pointed out above, dynamics of atom subsystem is simpler for modeling because of less quantity of freedom degrees. Binary and higher correlation functions naturally arise in considering superradiance in prolonged systems and related problems such as self-induced phase transitions [8]. Reduced description method opens the possibility to study the behavior of quasispin atom operator correlation functions of arbitrary orders via including them in RDPs. In [27], we developed these ideasand revealed characteristic problems of such approach. Indeed, RDM equations use averages of operators calculated with a quasi-equilibrium operator with a linear combination of RDPs in the exponent indicator. In the case of RDPs, which are linear in quasispin operator, the technique of specified calculations is well known, but it is problematic for more complicated operator forms.In this

paper, the development of our previous ideas of overcoming the corresponding difficulties through the consideration of the small deviation $\delta\eta_2 = \eta_2 - \eta_{20}$ of the RDP η_2 from its value at using only the linear form of spin operators. In the resent paper calculation Now the derivation of ρ_q in the perturbation theory by $\delta\eta_2$ is simplified. This theory is implemented in the technique of calculations with spin operators.

5. Conclusions

Theresultsofthispaperconsist in constructing the set of equations of the electrodynamics of continuous medium created by two-level emitters with random (or fixed) orientation. They have the usual Maxwell form if the medium charge and emitter current density are taken into account. The system evolution is investigated in the reduced description scheme providing the possibility of studying field correlations. Material equations connecting polarization and current density with electric and magnetic field parameters prove to be necessary. Such equations are built in a way taking into account the resonant nature of matter-field interaction. The possibility of using the complete electric field in equations is substantiated if the matter-field interaction is considered to be a small parameter. The local field characteristics depend on the emitter subsystem energy density. The evolution equation for this quantity should be derived in our next paper. Thus, presented results arethebasisforthe futureinvestigationof self-ordering processes in the Dicke model with the picture of field behavior.

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