BOGOLYUBOV REDUCED DESCRIPTION METHOD: SOME REMARKS AND PROVES

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Bogolyubov's reduced description method is based on his functional hypothesis. The reduced description of system's nonequilibrium state occurs over long-time scales $t >> \tau_0$. The synchronization time τ_0 determines the set of reduced description parameters $\{\eta_a(t)\}$, which fully describe the state of the system. The problem of finding the statistical operator of the system at the time of reduced description $\rho(\eta)$ and the effective initial values of the reduced description parameters $\{\eta_a(0)\}$ is posed and solved. This problem is solved only if there is a small parameter in the theory, with which the perturbation theory is built. The mathematical structure of the operators of the reduced description parameters $\{\hat{\eta}_a\}$ and the Hamilton operator of the system $\hat{H}$ plays the main role in this. The paper investigates the Peletminskii–Yatsenko model, in which $\hat{H} = \hat{H}_0 + \hat{H}_1$, $\hat{H}_0 \sim \lambda^0$, $\hat{H}_1 \sim \lambda^1$ ($\lambda << 1$), and the operators $\{\hat{H}_0, \hat{\eta}_a\}$ form a Lie algebra. The kinetics of the slow variable model, in which $[\hat{H}, \hat{\eta}_a] \sim \lambda^1$ ($\lambda << 1$), is also investigated. In these models, integral equations for the statistical operator $\rho(\eta)$ and the effective initial values of the reduced description parameters $\{\eta_a(0)\}$ are derived. Their solutions are investigated in the main and first orders of the perturbation theory for a small parameter λ. The right-hand sides of the time equations for the reduced description parameters $L_a(\eta)$ are investigated with accuracy up to the second-order contributions for λ. The paper thoroughly discusses the basic concepts of the reduced description method with remarks that supplement existing literature. In the Peletminskii–Yatsenko model, complex relations are proved that are given in the literature without justification or proved too complicatedly (in particular, this concerns the derivation of the integral equation for the initial values of the parameters $\{\eta_a(t)\}$). In our study of the model of slow variables, the operators of the reduced description parameters $\{\hat{\eta}_a\}$ are not specified, which leads to the implementation of the reduced description with rather complex algebra and operator analysis calculations. At the same time, a reasonable degree of details in the construction of the reduced description of the system is chosen.

Keywords: functional hypothesis, reduced description method, reduced description parameters, effective initial values, Peletminskii–Yatsenko model, slow variable kinetics model.

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1. Introduction

The theory of non-equilibrium processes is an important component of modern theoretical physics. The main approach of this theory – the method of reduced description of non-equilibrium systems – was developed by M. M. Bogolyubov in the 1940s. The systematic development of this method since the 1960s was carried out by S. V. Peletminskii, who, together with his students, investigated the general issues of the method and applied it to a number of systems and non-equilibrium processes in them. The history of the creation of this method can be traced to some extent in publications [1–9], although the list of which is actually incomplete. The research was largely aimed at improving the derivation of the Boltzmann kinetic equation and constructing gas dynamics based on it, taking into account dissipative processes. The monograph by M. M. Bogolyubov "Problems of Dynamic Theory in Statistical Physics" (1946), the main idea of which was, according to J. Uhlenbeck (1958), the idea of a functional hypothesis. Several researchers believe that its origin is associated with the method of S. Chapman and D. Enskog, although M. M. Bogolyubov himself (1946) linked it to his research on nonlinear mechanics. The final formulation of the functional hypothesis was developed by S. V. Peletminskii and summarized in his monograph with A. I. Akhiezer "Methods of Statistical Physics" (1977), where the task of proving it was set.

This article is based on the aforementioned studies and takes into account a number of the authors' work in connection with the further development of the theory, its applications, and its teaching to students. In particular, this concerns a special course at the PhD level "Principles of Physical Kinetics and Plasma Theory" and a special course at the Bachelor level "Computer Methods for Research of Nonlinear Physical Systems". The materials of the article were used in conducting online classes via the Internet. Therefore, the proof of various statements is presented in greater detail than is usually the case in educational literature.

The work has the following structure. The Introduction gives a brief overview of the creation of a method for the abbreviated description of non-equilibrium processes. Section 2 discusses the reduced Bogolyubov method and its implementation in the Peletminskii–Yatsenko model. Section 3 is devoted to the implementation of the kinetics of slow variables in the reduced description method. Sections 2, 3 consist of logical subsections, important fragments of the subsections are highlighted in bold italics.

2. Bogolyubov reduced description method (RDM) and its realization in Peleminskii–Yatsenko (PYa) model

2.1. Bogolyubov functional hypothesis

The non-equilibrium states of the system are described by the average values $\mathrm{Sp}\,\rho(t)\hat{\eta}_a$ of some parameters, where $\hat{\eta}_a$ are the operators of these parameters (a is the parameter number).

Our study of non-equilibrium states is based on Bogolyubov's RDM [6, 8] according to which the statistical operator (SO) of a non-equilibrium system $\rho(t)$ at large times $t >> \tau_0$ depends on time and the initial state of the system $\rho(t=0) \equiv \rho_0$ only through the averages of a limited number of parameters $\eta_a(t,\rho_0)$

$$\rho(t) \xrightarrow[t>>\tau_0]{} \rho(\eta(t,\rho_0)) \qquad (\mathrm{Sp}\,\rho(\eta) = 1, \quad \mathrm{Sp}\,\rho(\eta)\hat{\eta}_a = \eta_a) \tag{1.1}$$

They are called the reduced description parameters (RDPs) and defined by the formula

$$\mathrm{Sp}\,\rho(t)\hat{\eta}_a \xrightarrow[t>>\tau_0]{} \eta_a(t,\rho_0) \tag{1.2}$$

($\hat{\eta}_a$ are RDP operators). Relation (1.1) is named the functional hypothesis and value τ_0 is called the synchronization time. ***The set of RDPs*** $\{\hat{\eta}_a(t,\rho_0)\}$ ***is determined by the synchronization time*** τ_0 ***and the reduced description by these parameters is observed at times*** $t >> \tau_0$ ***.*** The arrows in formulas (1.1) and (1.2) indicate that their right parts are the asymptotics of the left parts. It is convenient to write the solution of the quantum evolution equation using the Liouville operator $\mathbf{L}$

$$\partial_t\rho(t) = \mathbf{L}\rho(t), \qquad \mathbf{L}\rho \equiv \frac{i}{\hbar}[\rho,\hat{H}]; \qquad \rho(t) = e^{t\mathbf{L}}\rho_0 \tag{1.3}$$

where $\hat{H}$ is the Hamilton operator. The leading statement of the RDM is that ***the* SO** $\rho(\eta(t,\rho_0))$ ***exactly satisfies the Liouville equation***

$$\partial_t \rho(\eta(t,\rho_0)) = \frac{i}{\hbar}[\rho(\eta(t,\rho_0)), \hat{H}] \tag{1.4}$$

for $t >> \tau_0$, and ***the parameters*** $\eta_a(t,\rho_0)$ ***exactly satisfy the time equation***

$$\partial_t \eta_a(t,\rho_0) = L_a(\eta(t,\rho_0)), \qquad L_a(\eta) \equiv \frac{i}{\hbar}\mathrm{Sp}\,\rho(\eta)[\hat{H}, \hat{\eta}_a] \tag{1.5}$$

for $t >> \tau_0$.

In RDM applications it is important to study the influence of the initial state of the system on its evolution [6, 8]. To consider the Cauchy problem, ***the solutions of equations* (1.4) *and* (1.5) *can be continued to*** $t = 0$, although they describe the system evolution only for $t >> \tau_0$. Herewith, the values of the functions $\eta_a(t,\rho_0)$ at $t = 0$ are called effective initial conditions.

Taking into account the expression (1.5) for the function $L_a(\eta)$, the SO $\rho(\eta)$ satisfies the nonlinear differential equation

$$\sum\nolimits_a \frac{\partial \rho(\eta)}{\partial \eta_a} L_a(\eta) = \frac{i}{\hbar}[\rho(\eta), \hat{H}], \tag{1.6}$$

which can be solved only approximately in some perturbation theory. The case of the Hamiltonian $\hat{H} = \hat{H}_0 + \hat{H}_1$ with the main $\hat{H}_0$ and small $\hat{H}_1$ parts is particularly important in the paper. Formally, $\hat{H} \sim \lambda^0$, $\hat{H}_1 \sim \lambda^1$ where λ is a small parameter.

2.2. Peletminskii–Yatsenko model of the reduced description

Consideration of non-equilibrium states of the system begins with the selection of RDPs and their operators $\hat{\eta}_a$. The previous development of the theory of non-equilibrium processes is the basis for this work. At the same time, the modern trend is to expand the set of RDPs by taking into account non-equilibrium correlations (fluctuations) of the studied RDPs.

A constructive approach to the selection of RDPs was proposed by Peletminskii and Yatsenko using the symmetries of the basic Hamiltonian $\hat{H}_0$. This led them (see the original paper [10], as well as [8]) to the PYa model, in which the RDPs operators satisfy the condition

$$[\hat{H}_0, \hat{\eta}_a] = \sum\nolimits_b c_{ab} \hat{\eta}_b, \tag{1.7}$$

where c_{ab} is a C-number matrix. Based on (1.7) Peletminskii and Yatsenko established the functional hypothesis in the basic approximation of the perturbation theory in $\hat{H}_1$ [8]

$$e^{t\mathbf{L}_0}\rho_0 \xrightarrow[t>>\tau_0]{} \rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}t}\mathrm{Sp}\rho_0\hat{\eta})) \tag{1.8}$$

where

$$\mathbf{L}_0\rho \equiv \frac{i}{\hbar}[\rho, \hat{H}_0], \qquad \mathbf{L}_0\hat{\eta}_a = -\frac{i}{\hbar}\sum\nolimits_b c_{ab}\hat{\eta}_b, \qquad e^{t\mathbf{L}_0 t}\rho_q(Z(\eta)) = \rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}t}\eta))) \tag{1.9}$$

(in terms of [8], the relation (1.8) is called ergodic, since it is connected with the ergodic hypothesis by the authors). ***The last formula* (1.9) *was not obtained in* [8], *although it should be fulfilled within the RDM framework and is used significantly by us further on.***

Formulas (1.7), (1.8) include the SO $\rho_q(Y)$

$$\rho_q(Y) = \exp\left\{\Omega(Y) - \sum\nolimits_a Y_a \hat{\eta}_a\right\}, \qquad \mathrm{Sp}\,\rho_q(Y) \equiv 1 \tag{1.10}$$

which is called the quasi-equilibrium SO (it is close to the equilibrium SO in certain cases). Functions $Z_a(\eta)$ in (1.8) are determined by the condition

$$\mathrm{Sp}\rho_q(Z(\eta))\hat{\eta}_a = \eta_a \,. \tag{1.11}$$

The formulas (1.9) show that these functions satisfy the condition

$$Z_a(\eta e^{\frac{i}{\hbar}ct}) = \sum\nolimits_b Z_b(\eta) e_{ba}^{-\frac{i}{\hbar}ct} \tag{1.12}$$

at arbitrary t since the equality $Y_1 = Y_2$ follows from $\rho_q(Y_1) = \rho_q(Y_2)$. **(1.12) *results in a certain restriction on the matrix* c_{ab} *and function* $Z_a(\eta)$. *It will be investigated further.***

2.3. Integral equation for the statistical operator $\rho(\eta)$ of PYa model and its detailed solution

Bogolyubov showed that the equation (1.6) for the statistical operator $\rho(\eta)$ is invariant with respect to time inversion and should be supplemented with ***a boundary condition that selects the physical direction of time*** [6]. As such a condition, the authors of the model selected the functional hypothesis (1.8) for SO $\rho_0 = \rho(\eta)$ taking here the form

$$e^{t\mathbf{L}_0}\rho(\eta) \xrightarrow[t \gg \tau_0]{} \rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}t}\eta)) \,. \tag{1.13}$$

(1.13) is written in terms of evolution in the physical direction of time. This relation made it possible to obtain a nonlinear integral equation from the nonlinear differential equation (1.6)

$$\rho(\eta) = \rho_q(Z(\eta)) + \int_{-\infty}^{0} d\tau e^{-\tau\mathbf{L}_0} \mathrm{f}(e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta) \,, \tag{1.14}$$

where denoted

$$\mathrm{f}(\eta) = \frac{i}{\hbar}\left[\rho(\eta), \hat{H}_1\right] - \sum\nolimits_a \frac{\partial\rho(\eta)}{\partial\eta_a} M_a(\eta) \,, \qquad M_a(\eta) = \frac{i}{\hbar}\mathrm{Sp}\,\rho(\eta)[\hat{H}_1, \hat{\eta}_a] \,. \tag{1.15}$$

In these terms the right-hand side of the time equation (1.5) for RDP $\eta_a(t,\rho_0)$ with considering (1.7) takes the form

$$L_a(\eta) = \frac{i}{\hbar}\sum_b c_{ab}\eta_b + M_a(\eta) \,. \tag{1.16}$$

To obtain integral equation (1.14), we rewrite boundary condition (1.13) in the form

$$e^{\tau\mathbf{L}_0}\rho(e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta) \xrightarrow[\tau \to +\infty]{} \rho_q(Z(\eta)) \,. \tag{1.17}$$

Formulas (1.6), (1.15), and (1.16) lead to the identity

$$\frac{\partial}{\partial\tau}e^{\tau\mathbf{L}_0}\rho(e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta)=e^{\tau\mathbf{L}_0}\left(\frac{i}{\hbar}[\rho(\eta),\hat{H}_0]-\sum_{a,b}\frac{\partial\rho(\eta)}{\partial\eta_b}\frac{i}{\hbar}c_{ba}\eta_a\right)_{\eta\to e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta}=$$
$$=e^{\tau\mathbf{L}_0}\left(-\frac{i}{\hbar}[\rho(\eta),\hat{H}_1]+\sum_a\frac{\partial\rho(\eta)}{\partial\eta_a}M_a(\eta)\right)_{\eta\to e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta}=-e^{\tau\mathbf{L}_0}\mathrm{f}(e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta). \tag{1.18}$$

Integrating both sides of this relation over τ in interval $(0,+\infty)$, considering (1.17), gives the necessary equation (1.14).

Equation (1.14) can be solved by an iterative procedure in perturbation theory by $\hat{H}_1$ because its integrand expression has the first order in this operator. For the SO $\rho(\eta)$ we have

$$\rho(\eta)=\sum_{n=0}^{\infty}\rho^{(n)}(\eta)\,,\qquad \rho^{(0)}(\eta)=\rho_q(Z(\eta))\,,$$
$$\rho^{(1)}(\eta)=\int_{-\infty}^{0}d\tau e^{-\mathbf{L}_0\tau}\left\{\frac{i}{\hbar}\left[\rho_q(Z(\eta)),\hat{H}_1\right]-\sum_a\frac{\partial\rho_q(Z(\eta))}{\partial\eta_a}M_a^{(1)}(\eta)\right\}_{\eta\to e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta}. \tag{1.19}$$

The right-hand side $L_a(\eta)$ of the time equation for RDP is given by the relations (1.14), (1.15), and formulas

$$M_a(\eta)=\sum_{n=1}^{\infty}M_a^{(n)}(\eta)\,,\qquad M_a^{(1)}(\eta)=\frac{i}{\hbar}\mathrm{Sp}\rho_q(Z(\eta))\left[\hat{H}_1,\hat{\eta}_a\right],$$
$$M_a^{(2)}(\eta)=\frac{i}{\hbar}\mathrm{Sp}\rho^{(1)}(\eta)\left[\hat{H}_1,\hat{\eta}_a\right]. \tag{1.20}$$

Let us calculate $M_a^{(2)}(\eta)$ using (1.8), (1.15), (1.19), and (1.20)

$$M_a^{(2)}(\eta)=\frac{i}{\hbar}\int_{-\infty}^{0}d\tau\mathrm{Sp}\left[\hat{H}_1,\hat{\eta}_a\right]e^{-\mathbf{L}_0\tau}\left\{\frac{i}{\hbar}\left[\rho_q(Z(\eta)),\hat{H}_1\right]-\sum_b\frac{\partial\rho_q(Z(\eta))}{\partial\eta_b}M_b^{(1)}(\eta)\right\}_{\eta\to e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta}=$$
$$=\int_{-\infty}^{0}d\tau\mathrm{Sp}\left[\hat{H}_1,\hat{\eta}_a\right]\left\{\frac{i^2}{\hbar^2}\left[\rho_q(Z(e^{-\frac{i}{\hbar}c\tau}\eta)),\hat{H}_t(\tau)\right]-\frac{i}{\hbar}\sum_b\frac{\partial\rho_q(Z(e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta))}{\partial\eta_b}M_b(\eta)\right\}_{\eta\to e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta}=$$
$$=-\frac{1}{\hbar^2}\int_{-\infty}^{0}d\tau\mathrm{Sp}\left[\hat{H}_1,\hat{\eta}_a\right]\left[\rho_q(Z(\eta)),\hat{H}_1(\tau)\right]-$$
$$-\int_{-\infty}^{0}d\tau\sum_b\left\{\frac{\partial\mathrm{Sp}\rho_q(Z(e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta))\left[\hat{H}_1,\hat{\eta}_a\right]}{\partial\eta_b}\mathrm{Sp}\rho_q(Z(\eta))[\hat{H}_1,\hat{\eta}_b]\right\}_{\eta\to e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta}= \tag{1.21}$$
$$=-\frac{1}{\hbar^2}\int_{-\infty}^{0}d\tau\mathrm{Sp}\left[\hat{H}_1,\hat{\eta}_a\right]\left[\rho_q(Z(\eta)),\hat{H}_1(\tau)\right]-$$
$$-\frac{\hbar}{i}\int_{-\infty}^{0}d\tau\sum_b\frac{\partial M_a^{(1)}(e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta)}{\partial\eta_b}_{\eta\to e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta}\mathrm{Sp}\rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta))[\hat{H}_1,\hat{\eta}_b].$$

The last term here can be simplified as

$$\sum_b \frac{\partial M_a^{(1)}(e^{-\frac{i}{\hbar}\mathbf{c}\tau}\eta)}{\partial\eta_b}\Bigg|_{\eta\to e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta} \mathrm{Sp}\,\rho_q(Z(e^{\frac{i}{\hbar}\mathbf{c}\tau}\eta))[\hat{H}_1,\hat{\eta}_b] =$$
$$= \sum_{b,d} \frac{\partial M_a^{(1)}(\eta)}{\partial\eta_d} e_{db}^{\frac{i}{\hbar}\mathbf{c}\tau}\,\mathrm{Sp}[\hat{H}_1,\hat{\eta}_b]e^{\mathbf{L}_0\tau}\rho_q(Z(\eta)) =$$
$$= \sum_{b,d} \frac{\partial M_a^{(1)}(\eta)}{\partial\eta_d} e_{db}^{\frac{i}{\hbar}\mathbf{c}\tau}\,\mathrm{Sp}\,\rho_q(Z(\eta))e^{-\mathbf{L}_0\tau}[\hat{H}_1,\hat{\eta}_b] =$$
$$= \sum_{b,d} \frac{\partial M_a^{(1)}(\eta)}{\partial\eta_d} e_{db}^{\frac{i}{\hbar}\mathbf{c}\tau}\,\mathrm{Sp}\,\rho_q(Z(\eta))[e^{-\mathbf{L}_0\tau}\hat{H}_1, e^{-\mathbf{L}_0\tau}\hat{\eta}_b] = \tag{1.22}$$
$$= \sum_{b,d,e} \frac{\partial M_a^{(1)}(\eta)}{\partial\eta_d} e_{db}^{\frac{i}{\hbar}\mathbf{c}\tau}\,\mathrm{Sp}\,\rho_q(Z(\eta))[\hat{H}_1(\tau), \sum_e e_{be}^{-\frac{i}{\hbar}\mathbf{c}\tau}\,\hat{\eta}_e] =$$
$$= \sum_b \frac{\partial M_a^{(1)}(\eta)}{\partial\eta_b} \mathrm{Sp}\,\rho_q(Z(\eta))[\hat{H}_1(\tau),\hat{\eta}_b].$$

That leads finally to the following expression for the second order contribution $M_a^{(2)}(\eta)$ to the right-hand side $L_a(\eta)$ of the time equation (1.5) for RDPs

$$M_a^{(2)}(\eta) = -\frac{1}{\hbar^2}\int_{-\infty}^{0} d\tau\,\mathrm{Sp}\rho_q(Z(\eta))\left[\hat{H}_1(\tau),[\hat{H}_1,\hat{\eta}_a] + i\hbar\sum\nolimits_b \hat{\eta}_b \frac{\partial M_a^{(1)}(Z(\eta))}{\partial\eta_b}\right] \tag{1.23}$$

given without proving in [8].

1.4. Improved derivation of the integral equation for effective initial conditions of the PYa model

Let us consider the calculation of effective initial conditions $\eta_a(0,\rho_0)$, somewhat simplifying the approach of [13]. From (1.7) we consistently have

$$\partial_t\,\mathrm{Sp}\,\rho(t)\hat{\eta}_a = \frac{i}{\hbar}\mathrm{Sp}\,\rho(t)[\hat{H},\hat{\eta}_a] = \frac{i}{\hbar}\sum_b c_{ab}\,\mathrm{Sp}\,\rho(t)\hat{\eta}_b + \frac{i}{\hbar}\mathrm{Sp}\,\rho(t)[\hat{H}_1,\hat{\eta}_a],$$
$$\partial_t \sum_b e_{ab}^{-\frac{i}{\hbar}\mathbf{c}t}\,\mathrm{Sp}\,\rho(t)\hat{\eta}_b = \sum_b e_{ab}^{-\frac{i}{\hbar}\mathbf{c}t}\,\frac{i}{\hbar}\mathrm{Sp}\,\rho(t)[\hat{H}_1,\hat{\eta}_b], \tag{1.24}$$
$$\sum_b e_{ab}^{-\frac{i}{\hbar}\mathbf{c}t}\,\mathrm{Sp}\,\rho(t)\hat{\eta}_b - \mathrm{Sp}\,\rho_0\hat{\eta}_a = \sum_b \int_0^t d\tau\, e_{ab}^{-\frac{i}{\hbar}\mathbf{c}\tau}\,\frac{i}{\hbar}\mathrm{Sp}\,\rho(\tau)[\hat{H}_1,\hat{\eta}_b].$$

According to (1.3) and (1.4), $\rho(t)$ and $\rho(\eta(t,\rho_0))$ satisfy the same time equation and therefore the following formulas are valid

$$\sum_b e_{ab}^{-\frac{i}{\hbar}\mathbf{c}t}\,\mathrm{Sp}\,\rho(\eta(t,\rho_0))\hat{\eta}_b - \mathrm{Sp}\,\rho(\eta(0,\rho_0))\hat{\eta}_a =$$
$$= \sum_b \int_0^t d\tau\, e_{ab}^{-\frac{i}{\hbar}\mathbf{c}\tau}\,\frac{i}{\hbar}\mathrm{Sp}\,\rho(\eta(\tau,\rho_0))[\hat{H}_1,\hat{\eta}_b], \tag{1.25}$$

$$\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\eta_b(t,\rho_0)-\eta_a(0,\rho_0)=\sum_b\int_0^t d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\frac{i}{\hbar}\mathrm{Sp}\rho(\eta(\tau,\rho_0))[\hat{H}_1,\hat{\eta}_b].$$

The last formula is analogous to the last one in (1.24). Their difference has the form

$$\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\mathrm{Sp}\rho(t)\hat{\eta}_b-\mathrm{Sp}\rho_0\hat{\eta}_a-\sum_b e^{-\frac{i}{\hbar}\mathbf{c}t}_{ab}\eta_b(t,\rho_0)+\eta_a(0,\rho_0)=$$
$$=\sum_b\int_0^t d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\frac{i}{\hbar}\mathrm{Sp}\{\rho(\tau)-\rho(\eta(\tau,\rho_0))\}[\hat{H}_1,\hat{\eta}_b].\tag{1.26}$$

In this relation, considering the functional hypothesis and the definition of the function $\eta_a(t,\rho_0)$, it is possible to go to the limit $t\to+\infty$, which gives

$$\eta_a(0,\rho_0)-\mathrm{Sp}\rho_0\hat{\eta}_a=\sum_b\int_0^{+\infty}d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\frac{i}{\hbar}\mathrm{Sp}\{\rho(\tau)-\rho(\eta(\tau,\rho_0))\}[\hat{H}_1,\hat{\eta}_b].\tag{1.27}$$

Taking into account formula (1.3), its analogue for $\rho(\eta(\tau,\rho_0))$, and the identity $\mathrm{Sp}(e^{t\mathbf{L}}\hat{a})\hat{b}=\mathrm{Sp}\,\hat{a}e^{-t\mathbf{L}}\hat{b}$, leads to the integral equation for $\eta_a(0,\rho_0)$

$$\eta_a(0,\rho_0)=\mathrm{Sp}\rho_0\hat{\eta}_a+\sum_b\int_0^{+\infty}d\tau e^{-\frac{i}{\hbar}\mathbf{c}\tau}_{ab}\frac{i}{\hbar}\mathrm{Sp}\{\rho_0-\rho(\eta(0,\rho_0))\}e^{-\tau\mathbf{L}}[\hat{H}_1,\hat{\eta}_b].\tag{1.28}$$

This integral equation was obtained in [8] ***by a more complicated procedure***. It is solved in the perturbation theory in interaction $\hat{H}_1$, which, in accordance with (1.18), gives

$$\eta_a(0,\rho_0)=\mathrm{Sp}\rho_0\hat{\eta}_a+$$
$$+\int_0^{+\infty}d\tau\frac{i}{\hbar}\mathrm{Sp}\{\rho_0-\rho_q(Z(\mathrm{Sp}\rho_0\hat{\eta}))\}[\hat{H}_1(\tau),\hat{\eta}_b]+O(\lambda^2)\qquad\left(\hat{H}_1(t)\equiv e^{-t\mathbf{L}_0}\hat{H}_1\right).\tag{1.29}$$

This formula makes it possible to investigate the Cauchy problem in the PYa model for an arbitrary initial state of the system described by SO ρ_0.

2. Bogolyubov RDM and its realization in kinetics of slow variables

2.1. Kinetics of slow variables in the RDM

Operators $\hat{\eta}_a$ of slow variables η_a satisfy the condition $[\hat{H},\hat{\eta}_a]\sim\lambda,\ \lambda<<1$ where $\hat{H}$ and λ are the Hamiltonian of the system and a small parameter. The Liouville operator $\mathbf{L}$ for an arbitrary operator $\hat{a}$ is defined by the expression $\mathbf{L}\hat{a}=\frac{i}{\hbar}[\hat{a},\hat{H}]$. A most important theory of slow variables is hydrodynamics, in which $\hat{\eta}_a=\hat{\zeta}_{nk}$ $(k\le k_0\sim\lambda<<1)$ where $\hat{\zeta}_{nk}$ is the Fourier transform of the integral equation density $\hat{\zeta}_n(x)$ for the considered system [12].

Complete kinetics of slow variables needs calculating the SO $\rho(\eta)$ and effective initial conditions $\eta_a(0,\rho_0)$. According to (1.1), (1.5), and (1.6) one must solve the set of equations

$$\sum_a \frac{\partial\rho(\eta)}{\partial\eta_a} L_a(\eta) = \mathbf{L}\rho(\eta)\,, \qquad L_a(\eta) = \mathrm{Sp}\,\hat{\eta}_a \mathbf{L}\rho(\eta)\,, \qquad \mathrm{Sp}\,\rho(\eta)\hat{\eta}_a = \eta_a\,. \tag{2.1}$$

The system considered is invariant with respect to time reversal. Therefore, one needs ad to (2.1) ***a boundary condition that accounts for time evolution in the physical direction of time***. With this purpose, here the functional hypothesis is taken. According to (1.1), it is given by $\rho(t) \xrightarrow[t\gg\tau_0]{} \rho(\eta(t,\rho_0))$ where $\rho(t) = e^{t\mathbf{L}}\rho_0$ and $\rho(\eta(t,\rho_0)) = e^{t\mathbf{L}}\rho(\eta(0,\rho_0))$ for $t \geq 0$ because of (1.3) and (1.4). Therefore (1.1) leads to useful asymptotic relations

$$e^{t\mathbf{L}}[\rho_0 - \rho(\eta(0,\rho_0))] \xrightarrow[t\gg\tau_0]{} 0\,, \qquad \lim_{\tau\to+\infty} e^{\tau\mathbf{L}}[\rho_0 - \rho(\eta(0,\rho_0))] = 0 \tag{2.2}$$

that can be used as the necessary ***boundary condition***. The last formula is written with τ instead of t to stress its mathematical sense without connection to time evolution. It is equivalent to the relation

$$\rho(\eta(0,\rho_0)) - \rho_0 = \int_0^{+\infty} d\tau e^{\tau\mathbf{L}}\mathbf{L}[\rho_0 - \rho(\eta(0,\rho_0))] \tag{2.3}$$

that can be considered as the ***boundary condition in the integral form***.

The only drawback of relations (2.3) is the presence of an arbitrary SO ρ_0 in it. In the theory of non-equilibrium processes, a significant role is played by the SO $\rho_q(Y)$ determined by the formula

$$\rho_q(Y) = e^{\Omega(Y) - \sum_a Y_a \hat{\eta}_a}\,, \qquad \mathrm{Sp}\,\rho_q(Y) = 1\,. \tag{2.4}$$

It includes microscopic variables $\hat{\eta}_a$ of the RDPs η_a, some parameters Y_a and the function $\Omega(Y)$ to be found from the normalization condition. This SO is often called quasi-equilibrium, since in a few cases it is close to the equilibrium SO. An important role is played by the effective initial conditions $\eta_a(0,\rho_q(Y))$, which correspond to the SO $\rho_q(Y)$, and the inverse function $Y_a(\eta)$ to the function $\eta_a(0,\rho_q(Y))$

$$\eta_a(0,\rho_q(Y(\eta))) = \eta_a\,, \qquad Y_a(\eta_a(0,\rho_q(Y))) = Y_a\,. \tag{2.5}$$

The first of these formulas allows us to rewrite the boundary condition (2.3) directly in terms of SO $\rho(\eta)$

$$\rho(\eta) - \rho_q(Y(\eta)) = \int_0^{+\infty} d\tau e^{\tau\mathbf{L}}\mathbf{L}[\rho_q(Y(\eta)) - \rho(\eta)] \tag{2.6}$$

that is more suitable for further applications.

2.2. Integral equations for SO $\rho(\eta)$ and effective initial conditions $\eta_a(0,\rho_0)$ in kinetics of slow variables and their solution

Considering the Liouville equation at the reduced description (2.1) in relation (2.6) we obtain

$$\rho(\eta) = \rho_q(Y(\eta)) + \int_0^{+\infty} d\tau e^{\tau\mathbf{L}}[\mathbf{L}\rho_q(Y(\eta)) - \sum_a \frac{\partial\rho(\eta)}{\partial\eta_a} L_a(\eta)] \,. \tag{2.7}$$

Equation (2.7) should be solved with taking into account the last two relations from (2.1) that give functions $Y_a(\eta)$ and $L_a(\eta)$. This formula can be considered as an integral equation for the SO $\rho(\eta)$ solvable by iterations in small parameter λ because

$$\begin{gathered} L_a(\eta) = \mathrm{Sp}\,\hat\eta_a \mathbf{L}\rho(\eta) = -\mathrm{Sp}(\mathbf{L}\hat\eta_a)\rho(\eta) \sim \lambda \qquad (\mathrm{Sp}(\mathbf{L}\hat A)\hat B = -\mathrm{Sp}\,\hat A \mathbf{L}\hat B\,); \\ \mathbf{L}\rho = \int_0^1 d\mu \rho^{1-\mu}(\mathbf{L}\ln\rho - \mathrm{Sp}\,\mathbf{L}\ln\rho)\rho^{\mu} \quad => \\ \mathbf{L}\rho_q(Y) = -\int_0^1 d\mu\, \rho_q(Y)^{1-\mu} \sum\nolimits_a Y_a(\mathbf{L}\hat\eta_a - \mathrm{Sp}\rho_q(Y)\mathbf{L}\hat\eta_a)\rho_q(Y)^{\mu} \sim \lambda \,. \end{gathered} \tag{2.8}$$

Let us multiply equation (2.3) by the microscopic value of RDP $\hat\eta_a$ and take Sp from the resulting expression. With the last formula (2.1) it gives the relation

$$\eta_a(0,\rho_0)) - \mathrm{Sp}\,\hat\eta_a\rho_0 = \int_0^{+\infty} d\tau\, \mathrm{Sp}\,\hat\eta_a e^{\tau\mathbf{L}}\mathbf{L}[\rho_0 - \rho(\eta(0,\rho_0))]$$

or equation

$$\eta_a(0,\rho_0) = \mathrm{Sp}\,\rho_0\hat\eta_a + \int_0^{+\infty} d\tau\, \mathrm{Sp}(\mathbf{L}\hat\eta_a) e^{\tau\mathbf{L}}[\rho(\eta(0,\rho_0)) - \rho_0] \,. \tag{2.9}$$

which can be considered as an integral equation for effective initial conditions $\eta_a(0,\rho_0)$ solvable by iteration over a small parameter λ.

The solution of the integral equation of (2.7) is investigated in the form of three expansions in powers of λ

$$\begin{gathered} \rho(\eta) = \rho^{(0)}(\eta) + \rho^{(1)}(\eta) + O(\lambda^2) , \qquad Y_a(\eta) = Y_a^{(0)}(\eta) + Y_a^{(1)}(\eta) + O(\lambda^2) , \\ L_a(\eta) = L_a^{(1)}(\eta) + L_a^{(2)}(\eta) + O(\lambda^3) \,. \end{gathered} \tag{2.10}$$

From equations (2.1) and (2.7) for the quantities $\rho(\eta)$ and $Y_a(\eta)$ in the main and first order of the perturbation theory, it is obtained

$$\begin{gathered} \rho^{(0)} = w , \qquad \rho^{(1)} = \rho_q(Y(\eta))^{(1)} + \int_0^{+\infty} d\tau e^{\tau\mathbf{L}} \left(\mathbf{L}w - \sum_a \frac{\partial w}{\partial\eta_a} L_a^{(1)} \right); \\ \mathrm{Sp}w\hat\eta_a = \eta_a , \qquad \mathrm{Sp}\rho^{(1)}\hat\eta_a = 0 \end{gathered} \tag{2.11}$$

where the designations

$$w = \rho_q(Z(\eta)) , \qquad Z_a(\eta) = Y_a^{(0)}(\eta) , \qquad \Phi(\eta) = \Omega(Z(\eta)) \tag{2.12}$$

were introduced. The main contributions to the right-hand sides of the equations for RDPs accordingly (2.1) have the form

$$L_a^{(1)} = \mathrm{Sp}\,w\dot{\hat\eta}_a , \qquad L_a^{(2)} = \mathrm{Sp}\,\rho^{(1)}\dot{\hat\eta}_a \qquad (\,\dot{\hat\eta}_a \equiv -\mathbf{L}\hat\eta_a). \tag{2.13}$$

Let us proceed to the calculation of the SO $\rho^{(1)}(\eta)$ from (2.11). The first contribution to it is given by

$$\rho_q(Y(\eta))^{(1)} = -\sum\nolimits_a Y_a^{(1)} \int_0^1 d\mu w^{1-\mu}(\hat{\eta}_a - \eta_a) w^{\mu} \tag{2.14}$$

taking into account for the formula

$$e^{\hat{A}+\hat{B}} = e^{\hat{A}}\left(1 + \int_0^1 d\mu e^{-\mu\hat{A}} \hat{B} e^{\mu\hat{A}} + O(\hat{B}^2)\right) \tag{2.15}$$

and expressions (2.4), (2.11), and (2.12) (in (2.15) $\hat{A}$ and $\hat{B}$ are arbitrary operators; ***about operator algebra and analysis calculations see our book*** [13]).

Under the integral in (2.11) the expressions are necessary

$$\mathbf{L}w = \int_0^1 d\mu w^{(1-\mu)}(\mathbf{L}\ln w - \mathrm{Sp} w\mathbf{L}\ln w) w^{\mu} = \sum\nolimits_a Z_a \int_0^1 d\mu w^{(1-\mu)}(\dot{\hat{\eta}}_a - \mathrm{Sp} w\dot{\hat{\eta}}_a) w^{\mu} \ ,$$

$$L_a^{(1)} = \mathrm{Sp} w\dot{\hat{\eta}}_a = -\mathrm{Sp} w\mathbf{L}\hat{\eta}_a = \mathrm{Sp}(\mathbf{L}w)\hat{\eta}_a = \tag{2.16}$$

$$= \mathrm{Sp}\sum\nolimits_b Z_b \int_0^1 d\mu\, w^{(1-\mu)}(\dot{\hat{\eta}}_b - \mathrm{Sp} w\dot{\hat{\eta}}_b) w^{\mu}\,\hat{\eta}_a = \sum\nolimits_b Z_b(\dot{\hat{\eta}}_b, \hat{\eta}_a)$$

where the symmetrical bilinear form

$$(\hat{A}, \hat{B}) = \int_0^1 d\mu\, \mathrm{Sp} w^{1-\mu}(\hat{A} - \mathrm{Sp} w\hat{A}) w^{\mu} \hat{B} \tag{2.17}$$

is introduced.

The multiplier $\dfrac{\partial w}{\partial \eta_a}$ in (2.11) at $L_a^{(1)}$ using the formulas

$$\frac{\partial w}{\partial \eta_a} = \sum\nolimits_b \left.\frac{\partial \rho_q(Y)}{\partial Y_b}\right|_{Y\to Z(\eta)} \frac{\partial Z_b(\eta)}{\partial \eta_a},$$

$$\frac{\partial}{\partial x} e^{\hat{A}(x)} = \int_0^1 d\mu e^{(1-\mu)\hat{A}(x)} \left(\frac{\partial}{\partial x} A(x)\right) e^{\mu\hat{A}(x)} \ \Rightarrow \left.\frac{\partial \rho_q}{\partial Y_a}\right|_{Y\to Z(\eta)} = -\int_0^1 d\mu w^{1-\mu}(\hat{\eta}_a - \eta_a) w^{\mu}; \tag{2.18}$$

$$\mathrm{Sp}\, w\hat{\eta}_a = \eta_a \ \Rightarrow \ \sum\nolimits_b (\hat{\eta}_b, \hat{\eta}_a) \frac{\partial Z_b}{\partial \eta_c} = \delta_{ac}$$

can be written as

$$\frac{\partial w}{\partial \eta_a} = -\sum\nolimits_b \int_0^1 d\mu w^{1-\mu}(\hat{\eta}_b - \eta_b) w^{\mu} (\hat{\eta}_b, \hat{\eta}_a)^{-1} \tag{2.19}$$

and therefore with (2.16)

$$\sum\nolimits_a \frac{\partial w}{\partial \eta_a} L_a^{(1)} = \sum_{a,b,c} \int_0^1 d\mu w^{1-\mu}(\hat{\eta}_b - \eta_b) w^{\mu} (\hat{\eta}_b, \hat{\eta}_a)^{-1} Z_c(\dot{\hat{\eta}}_c, \hat{\eta}_a)\,. \tag{2.20}$$

Now the integrand in (2.11), taking into account (2.16), takes on a compact form

$$\mathbf{L}w - \sum_a \frac{\partial w}{\partial \eta_a} L_a^{(1)} = \sum\nolimits_a Z_a \int_0^1 d\mu w^{1-\mu}(\mathbf{Q}\dot{\hat{\eta}}_a - \mathrm{Sp} w\mathbf{Q}\dot{\hat{\eta}}_a) w^{\mu} \tag{2.21}$$

where the operator **Q** (see [12]) is introduced by

$$\mathbf{Q}=1-\mathbf{P}\,,\qquad \mathbf{P}\hat{A}\equiv\sum_{a,b}(\hat{A},\hat{\eta}_a)(\hat{\eta}_a,\hat{\eta}_b)^{-1}\hat{\eta}_b\,. \tag{2.22}$$

The final expression for the SO $\rho^{(1)}(\eta)$ from (2.11) and (2.14) together with (2.21) has the form

$$\begin{aligned}\rho^{(1)}&=-\sum\nolimits_a Y_a^{(1)}\int_0^1 d\mu w^{1-\mu}(\hat{\eta}_a-\eta_a)w^{\mu}+\\&+\sum\nolimits_a Z_a\int_0^{+\infty}d\tau\,\int_0^1 d\mu\, e^{\tau\mathbf{L}}w^{1-\mu}(\mathbf{Q}\dot{\hat{\eta}}_a-\mathrm{Sp}w\mathbf{Q}\dot{\hat{\eta}}_a)w^{\mu}\,.\end{aligned} \tag{2.23}$$

The average value of the quantity A with SO (2.23) is given by the expression

$$\begin{gathered}\mathrm{Sp}\rho^{(1)}\hat{A}=-\sum\nolimits_a Y_a^{(1)}(\hat{\eta}_a,\hat{A})+\sum\nolimits_a Z_a\int_0^{+\infty}d\tau\,\,(\mathbf{Q}\dot{\hat{\eta}}_a,\hat{A}(\tau))\\ \big(\,\hat{A}(t)\equiv e^{-t\mathbf{L}}\hat{A}\,,\quad \mathrm{Sp}(e^{t\mathbf{L}}\hat{A})\hat{B}=\mathrm{Sp}\,\hat{A}e^{-t\mathbf{L}}\hat{B}\,\big)\end{gathered} \tag{2.24}$$

where bilinear forms (2.17) are used. The condition (2.11) $\mathrm{Sp}\rho^{(1)}(\eta)\hat{\eta}_a=0$ and formula (2.24) give expression for $Y_a^{(1)}$

$$Y_c^{(1)}=\sum\nolimits_a Z_a\int_0^{+\infty}d\tau\,\,(\mathbf{Q}\dot{\hat{\eta}}_a,\hat{\eta}_b(\tau))(\hat{\eta}_b,\hat{\eta}_c)^{-1}\,. \tag{2.25}$$

Substituting this expression into formula (2.23), we obtain

$$\mathrm{Sp}\rho^{(1)}\hat{A}=\sum\nolimits_a Z_a\int_0^{+\infty}d\tau\,\,(\mathbf{Q}\dot{\hat{\eta}}_a,\mathbf{Q}\hat{A}(\tau))\,. \tag{2.26}$$

From (2.13), (2.16) and taking into account formula (2.26), we sequentially obtain contributions of the first and second order in the small parameter of the theory λ to the time equation for the RDP kinetics of slow variables

$$\begin{aligned}L_a^{(1)}&=\mathrm{Sp}\,w\dot{\hat{\eta}}_a=\sum\nolimits_b Z_b(\dot{\hat{\eta}}_b,\hat{\eta}_a)\,,\\ L_a^{(2)}&=\mathrm{Sp}\,\rho^{(1)}\dot{\hat{\eta}}_a=\sum\nolimits_b Z_b\int_0^{+\infty}d\tau\,\,(\mathbf{Q}\dot{\hat{\eta}}_b,\mathbf{Q}\dot{\hat{\eta}}_a(\tau))\,.\end{aligned} \tag{2.27}$$

The effective initial conditions $\eta_a(0,\rho_0)$ for time equations (1.28) can be found from the integral equation (2.9) by perturbation theory in λ that gives

$$\begin{gathered}\eta_a(0,\rho_0)=\eta_a^{(0)}+\eta_a^{(1)}+O(\lambda^2)\,,\qquad \eta_a^{(0)}=\eta_{a0}\qquad \big(\,\eta_{a0}\equiv\mathrm{Sp}\rho_0\hat{\eta}_a\big);\\ \eta_a^{(1)}=\int_0^{+\infty}d\tau\mathrm{Sp}[\rho_0-w(\eta_0)]\dot{\hat{\eta}}_a(\tau)\end{gathered} \tag{2.28}$$

because according to (2.11) and (2.12) $\rho^{(0)}(\eta)=\rho_q(Z(\eta))=w(\eta)$. Further, formula $\mathrm{Sp}(\rho_0-w(\eta_0))\hat{\eta}_a=0$ with definition (2.22) of operator $\mathbf{P}$ gives the identity $\mathrm{Sp}(\rho_0-w(\eta_0))\mathbf{P}\hat{A}=0$ and allows us to rewrite $\eta_a^{(1)}$ in the form

$$\eta_a^{(1)}=\int_0^{+\infty}d\tau\mathrm{Sp}[\rho_0-w(\eta_0)]\mathbf{Q}\dot{\hat{\eta}}_a(\tau)\,. \tag{2.29}$$

Thus, the effective initial conditions $\eta_a(0,\rho_0)$ are calculated coherently.

3. Conclusions

The work is based on the Bogolyubov's method of reduced description of non-equilibrium states. Its basic ideas are carefully discussed with remarks that supplement the existing literature. The initial idea of the method is a functional hypothesis that briefly (by relatively small amount of parameters) describes the system at large times $t \gg \tau_0$ (τ_0 is a characteristic time). Attention is drawn to the need of taking into account the physical direction of evolution in time by using a boundary condition that violates the invariance relative to time. A useful idea is to formulate it as the functional hypothesis in a simple case of a quasi-equilibrium distribution. The fruitfulness of the idea of extending the values of reduced description parameters to non-physical times and introducing effective initial conditions is noted in the paper.

The paper investigates the implementation of the reduced description method in the Peletminskii–Yatsenko model and in the slow variables model. In these models, there is a small parameter, on the basis of which the perturbation theory is built for calculating the quantities introduced at the reduced description of the system (statistical operator, effective initial conditions). For them, the corresponding integral equations are obtained, which are solved by iterations.

In the Peletminskii–Yatsenko model, the contribution view of the second order of smallness to the time equations for the RDPs is proved rigorously, and a simple derivation of the integral equation for the effective initial conditions is proposed. In our study, the parameters of slow variables are not specified, which leads to significant complication in the implementation of the reduced description. In this case, we propose reasonable techniques for solving the problem referring to our book on additional questions of quantum mechanics including the innovative consideration of the kinetics of slow variables.

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