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MATERIAL EQUATIONS IN ELECTRODYNAMICS OF MEDIUM CONSISTING OF TWO-LEVEL EMITTERS

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The process of self-ordering in the famous Dicke model was studied in the framework of eliminating the boson variables. But the reduced description method enables us to obtain also the picture of electromagnetic field evolution provided field amplitudes and correlation functions are included into the number of reduced description parameters. In the Dicke Hamiltonian structure the interaction term includes the operators of emitter dipole moments or dipole moment density (polarization) since a spatial system is under consideration. Thus operator evolution equations are based on using such operators and their derivatives. The chain of evolution equations for averaged field amplitudes and binary correlation functions are obtained with using the statistical operator calculated in a perturbation theory in quasispin-photon interaction assumed to be small. The problem of chain decoupling does not arise since at any step we have a closed set of equations. The sets should be solved on the basis of material equations for current density and their generalizations for more complicated correlation functions. The way to constructing such equations and estimating the material parameters which are necessary for the numerical modeling of the development of correlations is discussed in the paper.

Keywords: Dicke model; reduced description; current; material equations; binary correlations, numerical modeling.

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1. Introduction

The picture of self-organization in the system of two-level emitters interacting via electromagnetic field is one of the most interesting examples of ordering in a nonequilibrium process. Such a phenomenon predicted by Dicke [1] attracted wide interest among physicists from the points of view of both the general theory of nonequilibrium systems and the practical possibilities to be opened by the phenomenon implementation. It was named superradiance (or superfluorescence) since excited emitters come to their ground state (i.e. equilibrium with surrounding medium) with producing a great pulse of electromagnetic radiation with intensity proportional to N^2 where N denotes the total quantity of emitters. Such generation is based on the cooperative spontaneous emission instead of the stimulated emission used in lasers and so provides the opportunity of coherent light generation in the frequency range where the laser mechanism has no prospects because of the absence of mirrors (X-rays, γ-rays). Since 70-ies of the past century such ideas were widely discussed including their military application. Superradiance implementation in experiment proved to be very difficult and was put into life [2] 20 years later than the Dicke's idea was put forward. It is interesting to compare the history of this phenomenon investigation in theory and experiment with the research into other collective quantum phenomena [3]. Nevertheless in the early 80-ies the consistent theory of the Dicke superfluorescence was developed [4] on the basis of the Bogolyubov method of eliminating boson variables [5]. It seems to be interesting not only to find the main parameters of a superradiant pulse, but also to obtain some description of field evolution at pulse formation. Such interest may be explained by the great progress of quantum optics and use of exotic states of electromagnetic field in modern experiments [6]. Field behavior can be analyzed by dint of the reduced description method [7] proceeding from the Bogolyubov's functional hypothesis [8]. The authors were the first to apply the method to the Dicke model [9, 10]. Then field parameters of reduced description were involved into consideration [11, 12] and evolution equations for them were constructed. The complexity of the problem made us to turn to the numerical analysis of the field evolution [13, 14]. Firstly, we discussed some kind of molecular dynamics modeling, but the necessity of taking into account the separation of resonant modes turned us to applying some continuous functions for field and emitter subsystem description. In any way, we deal with the equations for field amplitudes and correlation functions that have to be solved with using material equations according to the ideas of electrodynamics of continuous media. The problems of constructing material equations for a spatial Dicke model are discussed in the present paper.

The article has the following structure. Section 2 discusses the basic equations of the theory. A description of the properties of the physical quantities operators required in this section is given in [7]. Section 3 is devoted to the construction of perturbation theory to account for the weak interaction of the electromagnetic field and the environment. Section 4 presents the material equations to the Maxwell equations.

2. Basic equations of the theory

The kinetics of the electromagnetic field in the medium of atoms considered to be stationary and two-level are investigated. In the excited state, they have energy $\hbar\omega$ and an electric dipole moment $\vec{d}$. In the generalized quasispin Dike model, the Hamilton operator of the system can be written in the form

$$\hat{H} = \hat{H}_0 + \hat{H}_1, \quad \hat{H}_0 = \hat{H}_{\mathrm{f}} + \hat{H}_{\mathrm{m}}, \quad \hat{H}_{\mathrm{f}} = \sum_{k,\alpha} \hbar\omega_k c^{+}_{\alpha k} c_{\alpha k}, \quad \hat{H}_{\mathrm{m}} = \hbar\omega \sum_{1\le a\le N} \hat{r}_{az},$$
$$\hat{H}_1 = -\int dx \hat{E}^t_n(x)\hat{P}_n(x), \quad \hat{P}_n(x) \equiv \sum_a 2\hat{r}_{ax} d_{an} \delta(x - x_a) \tag{1}$$

where $\hat{H}_{\mathrm{f}}$ is the Hamiltonian of the free electromagnetic field ($\omega_k \equiv ck$), $\hat{r}_{an}$ and d_{an} are the quasispin operator and the dipole moment of the a-th atom. The vectors d_{an} for different atoms differ only in the orientation ($d_{an}d_{an} \equiv d^2$), which is described by the distribution function w_d. $\hat{P}_n(x)$ is the density operator of the electric dipole moment of the system. The interaction of atoms and the field is determined by the transverse part $\hat{E}^t_n(x)$ of the electric field operator $\hat{E}_n(x)$, since the dipole-dipole interaction of atoms is neglected. We consider the interaction of atoms and field as weak that allows us to consider the dipole moment d formally as a small parameter.

The nonequilibrium state of the field will be described by parameters η_a that include the average transverse electric and magnetic fields $\zeta_\mu : E^t_n(x), B_n(x)$ and their binary correlations $(\zeta_\mu, \zeta_{\mu'})$. The general definition of the binary correlation function of operators $\hat{a}$ and $\hat{b}$ is given by the formula with the statistical operator ρ of the system and their anticommutator

$$(a,b) = \mathrm{Sp}\,\rho\{\hat{a},\hat{b}\}/2 - \mathrm{Sp}\,\rho\hat{a}\,\mathrm{Sp}\,\rho\hat{b}. \tag{2}$$

The necessity of taking into account at least binary correlations of the field in describing its state is related to the nonexistence of statistical operator that describes the field only by its average quantities. The state of the medium will be described by the average density $\varepsilon(x)$ of its energy

$$\varepsilon(x) = \mathrm{Sp}\,\rho\hat{\varepsilon}(x), \quad \hat{\varepsilon}(x) = \sum_a \hbar\omega\hat{r}_{az}\delta(x - x_a). \tag{3}$$

(hereinafter, the mean value of the quantity a is denoted by the same letter as its operator $\hat{a}$). Thus, the values η_a and $\varepsilon(x)$ are the parameters of the reduced description of the state of the system.

When constructing system dynamics equations, it is convenient to use Maxwell equations in operator form, since for an arbitrary operator $\hat{a}$, taking into account the Liouville quantum equation for a statistical operator $\rho(t)$

$$\partial_t \,\mathrm{Sp}\,\rho(t)\hat{a} = \mathrm{Sp}\,\rho(t)\hat{\dot{a}}\,, \qquad \hat{\dot{a}} \equiv \frac{i}{\hbar}[\hat{H},\hat{a}]\,, \tag{4}$$

where $\hat{\dot{a}}$ is the velocity change operator of the quantity with the operator $\hat{a}$ (sometimes it is convenient to use the notation $\hat{\dot{a}} \equiv \partial_t \hat{a}$, although all our operators are taken in the Schrödinger picture). Maxwell's equations in the operator form take the expected view

$$\hat{\dot{B}}_n = -c\,\mathrm{rot}_n \hat{E}\,, \qquad \hat{\dot{E}}_n = c\,\mathrm{rot}_n \hat{B} - 4\pi \hat{I}_n\,, \quad \mathrm{div}\hat{B} = 0\,, \quad \mathrm{div}\hat{E} = 4\pi\hat{\rho} \tag{5}$$

if we introduce such definitions for the operators of the complete electric field $\hat{E}_n$ and densities of current $\hat{I}_n$ and charge $\hat{\rho}$ of the system

$$\begin{gathered}\hat{E}_n = \hat{E}^t_n - 4\pi\hat{P}_n\,, \qquad \hat{I}_n \equiv \hat{\dot{P}}_n\,, \qquad \hat{\rho} \equiv -\mathrm{div}\hat{P}\,, \\ \hat{I}_n = -\sum_a 2\omega d_{an}\hat{r}_{ay}\delta(x - x_a)\,.\end{gathered} \tag{6}$$

The same applies to the operator equation for the energy density of the medium

$$\hat{\dot{\varepsilon}}(x) = \hat{I}_n(x)\hat{E}^t_n(x)\,. \tag{7}$$

The Maxwell temporal operator equations in terms of fields $\hat{E}^t_n(x)$ and $\hat{B}_n(x)$ according to (5) and (6) have the form

$$\hat{\dot{B}}_n = -c\,\mathrm{rot}_n \hat{E}^t + 4\pi c\,\mathrm{rot}_n \hat{P}\,, \qquad \hat{\dot{E}}^t_n = c\,\mathrm{rot}_n \hat{B}\,, \tag{8}$$

or in compact notation $\hat{\zeta}_\mu : \quad \hat{E}^t_n(x)\,, \quad \hat{B}_n(x)$

$$\hat{\dot{\zeta}}_\mu = i\sum_{\mu'} \mathbf{c}_{\mu\mu'}\hat{\zeta}_{\mu'} + \hat{Q}_\mu\,, \qquad \hat{Q}_\mu : \quad 0\,; \quad 4\pi c\,\mathrm{rot}_n \hat{P} \tag{9}$$

where $\mathbf{c}_{\mu\mu'}$ is some numerical matrix. The corresponding equation for the averages has the same form

$$\partial_t \zeta_\mu = i\sum_{\mu'} \mathbf{c}_{\mu\mu'}\zeta_{\mu'} + Q_\mu\,, \qquad Q_\mu : \quad 0\,; \quad 4\pi c\,\mathrm{rot}_n P\,. \tag{10}$$

Further on it is convenient to use even more compact notations $\hat{\zeta}_{\mu_i} \equiv \hat{\zeta}_i\,, \ \sum_{\mu_i} ... \equiv \sum_i ...$, in which relation (10) takes the form

$$\partial_t \zeta_1 = i\sum_2 \mathbf{c}_{12}\zeta_2 + Q_1\,. \tag{11}$$

It is easy to get the relations from (11)

$$\begin{gathered}\partial_t\{\hat{\zeta}_1,\hat{\zeta}_2\} = i\sum_{1'} \mathbf{c}_{11'}\{\hat{\zeta}_{1'},\hat{\zeta}_2\} + i\sum_{2'} \mathbf{c}_{22'}\{\hat{\zeta}_1,\hat{\zeta}_{2'}\} + \{\hat{Q}_1,\hat{\zeta}_2\} + \{\hat{\zeta}_1,\hat{Q}_2\}\,, \\ \partial_t \zeta_1\zeta_2 = i\sum_{1'} \mathbf{c}_{11'}\zeta_{1'}\zeta_2 + i\sum_{2'} \mathbf{c}_{22'}\zeta_1\zeta_{2'} + Q_1\zeta_2 + \zeta_1 Q_2\,,\end{gathered} \tag{12}$$

from which the evolution equation follows for the binary correlations defined in (2)

$$\partial_t(\zeta_1,\zeta_2) = i\sum_{1'}\mathbf{c}_{11'}(\zeta_{1'},\zeta_2) + i\sum_{2'}\mathbf{c}_{22'}(\zeta_1,\zeta_{2'}) + (Q_1,\zeta_2) + (\zeta_1,Q_2)\,. \tag{13}$$

Equations (11) and (13) give a complete set of time equations for the parameters η_a of the reduced description of the electromagnetic field in the medium, i.e. ζ_μ and $(\zeta_\mu,\zeta_{\mu'})$. The evolution equation for the energy density of the medium according to (7), taking into account definition (2), is

$$\partial_t\varepsilon(x) = L(x,\eta,\varepsilon)\,, \qquad L(x,\eta,\varepsilon) \equiv \mathrm{Sp}\,\rho(\eta,\varepsilon)\hat{I}_n(x)\hat{E}_n^t(x)\,. \tag{14}$$

Our consideration of nonequilibrium states of the system is based on the Bogolyubov idea of the functional hypothesis

$$\rho(t) \xrightarrow[t>>\tau_0]{} \rho(\eta(t),\varepsilon(t))\,,$$
$$\mathrm{Sp}\,\rho(\eta,\varepsilon)\hat{\eta}_a \equiv \eta_a\,, \qquad \mathrm{Sp}\,\rho(\eta,\varepsilon)\hat{\varepsilon}(x) \equiv \varepsilon(x)\,, \tag{15}$$

which is the basis of his method of reduced description of nonequilibrium processes (τ_0 is characteristic time, which depends on the initial state of the system). The statistical operator $\rho(\eta(t),\varepsilon(t))$ is the exact solution of the quantum Liouville equation

$$\partial_t\rho(\eta(t),\varepsilon(t)) = -\frac{i}{\hbar}[\hat{H},\rho(\eta(t),\varepsilon(t))]\,. \tag{16}$$

The study of nonequilibrium processes in the system is simplified by the presence of relations

$$\mathbf{L}_0\hat{\eta}_a = -i\sum_b \mathbf{c}_{ab}\hat{\eta}_b\,, \qquad \mathbf{L}_0\hat{\varepsilon}(x) = 0 \qquad (\mathbf{L}_0\hat{a} \equiv -\frac{i}{\hbar}[\hat{H}_0,\hat{a}]) \tag{17}$$

where $\mathbf{c}_{ab}$ is some numerical matrix, which is expressed through the matrix $\mathbf{c}_{\mu\mu'}$. These relations are a special case of the Peletminsky–Yatsenko model of the nonequilibrium process [7]. In this model, the statistical operator $\rho(\eta,\varepsilon)$ of the system satisfies the integral equation

$$\rho(\eta,\varepsilon) = \rho_q(\eta,\varepsilon) + \int_0^{+\infty} d\tau e^{\tau\mathbf{L}_0}\Big\{\mathbf{L}_1\rho(\eta,\varepsilon) - \sum_a \frac{\partial\rho(\eta,\varepsilon)}{\partial\eta_a}L_a(\eta,\varepsilon) -$$
$$-\int dx\frac{\delta\rho(\eta,\varepsilon)}{\delta\varepsilon(x)}L(x,\eta,\varepsilon)\Big\}_{\eta_a\to\sum_b e^{-i\mathbf{c}\tau}_{ab}\eta_b}\,. \tag{18}$$

Functions $L_a(\eta,\varepsilon)$, $L(x,\eta,\varepsilon)$ determine the evolution equations (10), (13) and (14) for the parameters of the reduced description of the system

$$L_a(\eta,\varepsilon) \;\equiv\; Q_1, \quad (Q_1,\zeta_2) + (\zeta_1,Q_2)\,, \tag{19}$$

where the averages are calculated using the statistical operator $\rho(\eta,\varepsilon)$.

The statistical operator $\rho_q(\eta,\varepsilon)$, according to the Peletminsky–Yatsenko model, looks like

$$\rho_q(\eta,\varepsilon) = \rho_{\mathrm{f}}(\eta)\rho_{\mathrm{m}}(\varepsilon)\,, \qquad \rho_{\mathrm{m}}(\varepsilon) = w_{\mathrm{r}}(\varepsilon)w_{\mathrm{d}}\,; \tag{20}$$

$$\rho_{\mathrm{f}}(\eta) = \exp\{\Omega(\eta) - \sum_a Z_a(\eta)\hat{\eta}_a\}\,, \qquad \mathrm{Sp}_{\mathrm{f}}\,\rho_{\mathrm{f}}(\eta) = 1\,, \quad \mathrm{Sp}_{\mathrm{f}}\,\rho_{\mathrm{f}}(\eta)\hat{\eta}_a = \eta_a\,;$$

$$w_{\mathrm{r}}(\varepsilon) = \exp\{\Phi(\varepsilon) - \int dx Z(x,\varepsilon)\hat{\varepsilon}(x)\}\,, \qquad \mathrm{Sp}_{\mathrm{r}}\,w_{\mathrm{r}}(\varepsilon) = 1\,, \qquad \mathrm{Sp}_{\mathrm{r}}\,w_{\mathrm{r}}(\varepsilon)\hat{\varepsilon}(x) = \varepsilon(x)\,;$$

$$\mathrm{Sp}_{\mathrm{d}}\,w_{\mathrm{d}} = 1\,; \qquad \mathrm{Sp}\ldots = \mathrm{Sp}_{\mathrm{f}}\,\mathrm{Sp}_{\mathrm{m}}\ldots\,, \qquad \mathrm{Sp}_{\mathrm{m}}\ldots = \mathrm{Sp}_{\mathrm{r}}\,\mathrm{Sp}_{\mathrm{d}}\ldots$$

Here $\rho_{\mathrm{f}}(\eta)$ is a quasi-equilibrium statistical operator of an electromagnetic field. The operator $\hat{\eta}_a$ in $\rho_{\mathrm{f}}(\eta)$ includes field operators $\hat{E}_n^t$, $\hat{B}_n$ and all their anticommutators. Therefore, in the exponent $\rho_{\mathrm{f}}(\eta)$ there is a quadratic form of the Bose-operators of the field, which ensures the existence of traces with it. The operator $w_{\mathrm{r}}(\varepsilon)$ is a locally equilibrium statistical operator of the medium and hence $Z(x,\varepsilon) \equiv T(x,\varepsilon)^{-1}$ is the inverse temperature of the medium. Traces with the operator $w_{\mathrm{r}}(\varepsilon)$ are taken in quasi-spin space. Value w_{d} is a function of the distribution of the orientations of the dipoles of the atoms of the medium.

3. Construction of perturbation theory for electromagnetic field and medium

Integral equation (18) should be solved with respect to the statistical operator $\rho(\eta,\varepsilon)$ in the perturbation theory on the small interaction of the electromagnetic field with the medium described by the Hamilton operator $\hat{H}_1$. It is convenient to assume that the formal small parameter is the dipole moment d of an atom, since $\hat{H}_1$ is proportional to *d*. According to (18),

$$\rho(\eta,\varepsilon) = \rho^{(0)} + \rho^{(1)} + O(d^2)\,, \qquad \rho^{(0)} = \rho_q \tag{21}$$

($a^{(s)}$ is the contribution of the order d^s to the value a). Now we show that the first order contributions to $L_a(\eta,\varepsilon)$ and $L(x,\eta,\varepsilon)$ are absent

$$L_a^{(1)} = 0\,, \qquad L^{(1)} = 0\,. \tag{22}$$

To this end, it should be noted that in further all traces that need to be calculated have a structure

$$\mathrm{Sp}\,\hat{a}_{\mathrm{f}}\hat{b}_{\mathrm{m}} = \mathrm{Sp}_{\mathrm{f}}\,\mathrm{Sp}_{\mathrm{m}}\,\hat{a}_{\mathrm{f}}\hat{b}_{\mathrm{m}} = \mathrm{Sp}_{\mathrm{f}}\,\hat{a}_{\mathrm{f}}\,\mathrm{Sp}_{\mathrm{m}}\,\hat{b}_{\mathrm{m}} \tag{23}$$

where operators $\hat{a}_{\mathrm{f}}$, $\hat{b}_{\mathrm{m}}$ refer to the electromagnetic field and environment. According to (6), (9), and (21), in the basic approximation averages (19) are calculated with the statistical operator ρ_q and the averaged values include the operators $\hat{r}_{ax}$ and $\hat{r}_{ay}$ linearly. The relations (22) become obvious since the formulas are true

$$\mathrm{Sp}_{\mathrm{r}}\,w_{\mathrm{r}}\hat{r}_{ax} = 0\,, \qquad \mathrm{Sp}_{\mathrm{r}}\,w_{\mathrm{r}}\hat{r}_{ay} = 0\,. \tag{24}$$

The averages of the product of several quasispin operators $\hat{r}_{al}$ with a statistical operator are easily calculated in the representation of these operators using Pauli matrices. For the calculation of such averages, a theorem of the Wick–Bloch–de Dominicis theorem type is valid (see, for example, [10]), but we will not present it here.

Even easier on the basis of (24), taking into account the expressions for the dipole moment density (1) $\hat{P}_n(x)$ and the current density (6) $\hat{I}_n(x)$ of the system, the absence of first order contributions in their mean values is proved.

$$P_n^{(1)} = 0\,, \qquad I_n^{(1)} = 0\,. \tag{25}$$

The first-order contribution $\rho^{(1)}$ to the statistical operator $\rho(\eta,\varepsilon)$ from integral equation (18) considering (1) and (20) – (22) is reduced to the form

$$\rho^{(1)} = -\frac{i}{\hbar}\int_{-\infty}^{0} d\tau \left[\rho_q , e^{\frac{i}{\hbar}\tau\hat{H}_0} \hat{H}_1 e^{-\frac{i}{\hbar}\tau\hat{H}_0} \right] = $$
$$= -\frac{i}{\hbar}\int_{-\infty}^{0} d\tau \int dx \left[\rho_{\mathrm{f}}\rho_{\mathrm{m}} , \hat{E}_n^t(x,\tau)\hat{P}_n(x,\tau) \right] \tag{26}$$

where indicated

$$\hat{E}_n^t(x,\tau) \equiv e^{\frac{i}{\hbar}\tau\hat{H}_{\mathrm{f}}} \hat{E}_n^t(x) e^{-\frac{i}{\hbar}\tau\hat{H}_{\mathrm{f}}}\,, \qquad \hat{P}_n(x,\tau) \equiv e^{\frac{i}{\hbar}\tau\hat{H}_{\mathrm{m}}} \hat{P}_n(x) e^{-\frac{i}{\hbar}\tau\hat{H}_{\mathrm{m}}} \tag{27}$$

Herewith the identity that is valid for the statistical operator $\rho_q(\eta,\varepsilon)$ in the Peletminsky–Yatsenko model is used [7]

$$e^{\frac{i}{\hbar}\tau\hat{H}_0} \rho_q(\eta,\varepsilon) e^{-\frac{i}{\hbar}\tau\hat{H}_0} = \rho_q(e^{-i\mathrm{c}\tau}\eta,\varepsilon) \tag{28}$$

Via the time equation method for quantities $\hat{P}_n(x,\tau)$ and $\hat{E}_n^t(x,\tau)$, taking into account the corresponding commutation relations for operators, we find

$$\hat{P}_n(x,\tau) \equiv \hat{P}_n(x)\cos\omega\tau + \hat{I}_n(x)\omega^{-1}\sin\omega\tau\,,$$
$$\hat{E}_{nk}^t(\tau) = \hat{E}_{nk}^t \cos\omega_k\tau + c\hat{Z}_{nk}\omega_k^{-1}\sin\omega_k\tau \qquad (\hat{Z}_n(x) \equiv \mathrm{rot}_n \hat{B}(x)). \tag{29}$$

The periodic boundary conditions and the corresponding Fourier transform definition are used here and hereafter

$$\mathrm{f}(x) = \frac{1}{V}\sum_k \mathrm{f}_k e^{ikx}\,, \qquad \mathrm{f}_k \equiv \int_V dx\, \mathrm{f}(x) e^{-ikx}\,. \tag{30}$$

4. Material equations to the Maxwell equations in the medium

Let us proceed to the calculation of the average current I_n. Formulas (4) and (6) show that the mean values of current and polarization are related by a simple formula

$$I_n = \partial_t P_n \tag{31}$$

and so it is convenient to start by calculating the average polarization. According to (23) and (26) we have

$$P_n^{(2)}(x) = \frac{i}{\hbar}\int_{-\infty}^{0} d\tau \int dx\, \mathrm{Sp}_{\mathrm{f}}\, \rho_{\mathrm{f}} \hat{E}_l^t(x',\tau)\, \mathrm{Sp}_{\mathrm{m}}\, \rho_{\mathrm{m}} \left[\hat{P}_n(x), \hat{P}_n(x',\tau) \right] \tag{32}$$

where with taking into account (1), (6), and (29)

$$\left[\hat{P}_n(x), \hat{P}_n(x',\tau) \right] = -4i\delta(x-x')\hat{a}_{nl}(x)\sin\omega\tau\,, \qquad \hat{a}_{nl}(x) \equiv \sum_a d_{an} d_{al} \hat{r}_{az} \delta(x - x_a)\,. \tag{33}$$

In terms of Fourier images, the mean polarization value according to (29) and (30) takes the form

$$P_n^{(2)}(x) = \frac{4}{\hbar V}\sum_k e^{ikx} \int_{-\infty}^{0} d\tau \left(E_{nk}^t \cos\omega_k\tau + cZ_{nk}\omega_k^{-1}\sin\omega_k\tau \right) a_{nl}(x)\sin\omega\tau \tag{34}$$

where

$$a_{nl}(x) \equiv \mathrm{Sp_m}\, \rho_\mathrm{m} \hat{a}_{nl}(x) = \mathrm{Sp_r}\, w_\mathrm{r} \mathrm{Sp_d}\, w_\mathrm{d} \hat{a}_{nl}(x)\,, \quad \mathrm{Sp_f}\, \rho_\mathrm{f} \hat{E}^t_{nk} = E^t_{nk}\,, \quad \mathrm{Sp_f}\, \rho_\mathrm{f} \hat{Z}_{nk} = Z_{nk}\,. \qquad (35)$$

The last two formulas take into account the third formula in (20), definitions (29) and additionally state that the mean values of the field are calculated directly with the statistical operator ρ_f.

In the expression for the function $a_{nl}(x)$, there is averaging over the orientations of atom dipoles of the medium. To simplify the consideration, in further we limit ourselves by the case of isotropic distribution of dipoles of atoms and the assumption that there are no correlations between them, which is natural, since in our consideration we neglect the dipole-dipole interaction of atoms. As a result, we have

$$\mathrm{Sp_d}\, d_{an} = 0\,, \quad \mathrm{Sp_d}\, w_\mathrm{d} d_{an} d_{bl} = \delta_{ab}\delta_{nl} d^2/3 \qquad (d_{an}d_{an} = d^2) \qquad (36)$$

(atom dipoles differ only in direction). In this case, according to (3) and (20) we have a simple expression for the function $a_{nl}(x)$ through the energy density of the medium.

$$a_{nl}(x) = \frac{d^2}{3\hbar\omega}\varepsilon(x)\delta_{nl}\,, \qquad \varepsilon(x) = \mathrm{Sp_r}\, w_\mathrm{r} \hat{\varepsilon}(x) \qquad (37)$$

(see (20)).

The integrals over τ in (34) should be taken in the class of generalized functions by associating their computations with the thermodynamic limit transition by the usual rule

$$\sum_k \ldots \overset{TL}{=} \frac{V}{(2\pi)^3}\int d^3p \ldots \qquad (38)$$

Regularizing the necessary integrals, we have

$$\begin{aligned} &\int_{-\infty}^{0} d\tau \cos\omega_k\tau \sin\omega\tau = \lim_{\varepsilon\to+0}\int_{-\infty}^{0} d\tau e^{\varepsilon\tau}\cos\omega_k\tau\sin\omega\tau = \\ &= \lim_{\varepsilon\to+0} \mathrm{Im}\frac{1}{2}\{[i(\omega+\omega_k)+\varepsilon]^{-1} + [i(\omega-\omega_k)+\varepsilon]^{-1}\} = \mathrm{P}\frac{\omega}{\omega_k^2-\omega^2} \end{aligned} \qquad (39)$$

since

$$\lim_{\varepsilon\to+0}(x+i\varepsilon)^{-1} = (x+i0)^{-1} = \mathrm{P}\frac{1}{x} - i\pi\delta(x)\,. \qquad (40)$$

In (39) and (40), the limit transitions should be understood in a weak sense. Practically, this means that these transitions should be performed after the thermodynamic limit transition in accordance with (38) and the calculation of the corresponding integrals (as usual, the symbol $\mathrm{P}\frac{1}{x}$ means that the integral of $\frac{1}{x}$ is taken in the principal value sense). Similarly to (39), we have

$$\begin{aligned} &\int_{-\infty}^{0} d\tau \sin\omega_k\tau \sin\omega\tau = \lim_{\varepsilon\to+0}\int_{-\infty}^{0} d\tau e^{\varepsilon\tau}\sin\omega_k\tau\sin\omega\tau = \\ &= \lim_{\varepsilon\to+0} \mathrm{Im}\frac{1}{2i}\{[i(\omega+\omega_k)+\varepsilon]^{-1} - [i(\omega-\omega_k)+\varepsilon]^{-1}\} = \frac{\pi}{2}\delta(\omega-\omega_k) \qquad (\omega,\omega_k > 0)\,. \end{aligned} \qquad (41)$$

As a result, we obtain the final expression for the mean polarization of the medium

$$P_{nk}^{(2)} = \frac{1}{V}\sum_{k'}[\kappa(k,k',\varepsilon)E_{nk'}^{t} + c\lambda(k,k',\varepsilon)Z_{nk'}] \tag{42}$$

where designated

$$\kappa(k,k',\varepsilon) = -\frac{4d^2}{3\omega\hbar^2}\varepsilon_{k-k'}\mathrm{P}\frac{\omega}{\omega^2-\omega_{k'}^2}, \qquad \lambda(k,k',\varepsilon) = \frac{2\pi d^2}{3\omega^2\hbar^2}\varepsilon_{k-k'}\delta(\omega-\omega_{k'}). \tag{43}$$

In its sense, the expression (42) for the polarization of the medium is the material equation of electrodynamics in the case of a spatially inhomogeneous medium. Other options for writing this material equation are possible, based on the identities of the type

$$\mathrm{f}_{nk} \equiv \frac{1}{V}\sum_{k'}\kappa(k,k',\varepsilon)E_{nk'}^{t}, \qquad \kappa(k,k',\varepsilon) \equiv \varepsilon_{k-k'}\chi_{k'}$$

$$\mathrm{f}_n(x) = \frac{1}{V}\sum_{k}e^{ikx}\kappa(k,\varepsilon(x))E_{nk}^{t} = \int_V dx'\kappa(x-x',\varepsilon(x))E_n^t(x'), \qquad \kappa(k,\varepsilon) = \varepsilon\chi_k \tag{44}$$

where f_{nk}, E_{nk}^t, $\kappa(k,\varepsilon)$ are Fourier images of functions $\mathrm{f}_n(x)$, $E_n^t(x)$, $\kappa(x,\varepsilon)$ determined according to (30). Regarding the expression for the polarization of the medium it gives

$$P_n^{(2)}(x) = \frac{1}{V}\sum_k e^{ikx}[\kappa(k,\varepsilon(x))E_{nk}^t + c\lambda(k,\varepsilon(x))Z_{nk}],$$

$$P_n^{(2)}(x) = \int_V dx'[k(x-x',\varepsilon(x))E_n^t(x') + c\lambda(x-x',\varepsilon(x))Z_n(x')]; \tag{45}$$

$$\kappa(k,\varepsilon) = -\frac{4d^2}{3\omega\hbar^2}\varepsilon\mathrm{P}\frac{\omega}{\omega^2-\omega_k^2}, \qquad \lambda(k,\varepsilon) = \frac{2\pi d^2}{3\omega^2\hbar^2}\varepsilon\delta(\omega-\omega_k)$$

The second version of the material equation is more natural because it simply takes into account the spatial inhomogeneity of the medium. The material coefficient $\kappa(k,\varepsilon)$ has the meaning of dielectric susceptibility. The coefficient $\lambda(k,\varepsilon)$ describes the effect of frequency dispersion in our terms, since $Z_n = \mathrm{rot}_n B = c^{-1}\partial_t E_n^t$ (see (8), (29)).

To calculate the mean current, we proceed from the formula (31), which gives

$$I_{nk}^{(2)} = (\partial_t P_{nk}^{(2)})^{(2)}. \tag{46}$$

It shows that the energy density ε should not be differentiated in (42) and (44) when calculating the current, since, according to (14), (19), and (22), $\partial_t\varepsilon \sim d^2$. In view of formulas (8), (24), and (29) we have

$$\partial_t E_n^t = cZ_n, \qquad \partial Z_n = -c\,\mathrm{rot}_n\mathrm{rot}\,E^t + O(d^2) \tag{47}$$

and therefore

$$I_n^{(2)}(x) = \frac{1}{V}\sum_k e^{ikx}[\sigma(k,\varepsilon(x))E_{nk}^t + c\xi(k,\varepsilon(x))Z_{nk}],$$

$$I_n^{(2)}(x) = \int_V dx'[\sigma(x-x',\varepsilon(x))E_n^t(x') + c\xi(x-x',\varepsilon(x))Z_n(x')] \tag{48}$$

where

$$\sigma(k,\varepsilon) \equiv -\lambda(k,\varepsilon)\omega_k^2 = -\frac{2\pi d^2}{3\hbar^2}\varepsilon\,\delta(\omega-\omega_k)\,,$$
$$\xi(k,\varepsilon) \equiv \kappa(k,\varepsilon) = -\frac{4d^2}{3\hbar^2}\varepsilon\,\mathrm{P}\frac{1}{\omega^2-\omega_k^2} \tag{49}$$

In its sense the expressions (48) for the current in the medium are material equations of electrodynamics in the case of a spatially inhomogeneous medium. The material coefficient $\sigma(k,\varepsilon)$ makes sense of conductivity. The material coefficient $\xi(k,\varepsilon)$ describes the effect of frequency dispersion in our terms, since $Z_n = \mathrm{rot}_n B = c^{-1}\partial_t E_n^t$ (see (8) and (29)).

According to (6) and (24), the average charge density of the medium is given by formulas

$$\rho = -\mathrm{div}P\,; \qquad \rho_k^{(1)} = 0\,, \qquad \rho_k^{(2)} = -ik_n P_{nk}^{(2)} \tag{50}$$

with taking into account expressions (42) and (45) for polarization. It should be noted that in the space-homogeneous state the charge of the medium is absent, since in this case the polarization vector has only a transverse component (in accordance with (29), $Z_{nk} = i[k,B_k]_n$).

The equations of electrodynamics in the medium, according to (4) and (5), have the usual form

$$\partial_t B_n = -c\,\mathrm{rot}_n E\,, \quad \partial_t E_n = c\,\mathrm{rot}_n B - 4\pi I_n\,, \quad \mathrm{div}B = 0\,, \quad \mathrm{div}E = 4\pi\rho\,. \tag{51}$$

Formulas (47) and (50) give material equations for them. With the considered accuracy, it is possible to replace the transverse field with a complete one in these equations since from (6) and (25) it follows that $E_n = E_n^t - 4\pi P_n = E_n^t + O(d^2)$. As a result, in our approach, the electromagnetic field in the medium is described by the average E_n, B_n and their binary correlations. Although we construct the reduced description on the basis of the Peletminsky–Yatsenko model, where mean fields E_n^t, B_n and their binary correlations are used as reduced description parameters, equations (51) remain accurate and the material equations with the accepted accuracy contain a complete electric field E_n. Note that in the considered approximation the material equations to the Maxwell equations do not include field correlations.

5. Conclusions

The results of this paper consist in constructing the set of equations of the electrodynamics of continuous medium created by two-level emitters with random (or fixed) orientation. They have usual Maxwell form if medium charge and emitter current density are taken into account. The system evolution is investigated in the reduced description scheme providing the possibility of studying field correlations. Material equations connecting the polarization and current density with electric and magnetic field parameters prove to be necessary. Such equations are built in the way taking into account the resonant nature of matter-field interaction. The possibility of using the complete electric field in equations is substantiated if the matter-field interaction is considered to be a small parameter. The local field characteristics depend on the emitter subsystem energy density. The evolution equation for this quantity should be derived in our next paper. Thus, presented results are the basis for the future investigation of self-ordering processes in the Dicke model with the picture of field behavior.

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